
Exploring a unified expression of many types of philosophies and structures

Takaaki Fujita^{1*}

¹ Independent Researcher, Tokyo, Japan.

ORCID: <https://orcid.org/0009-0007-1509-2728>

Email: Takaaki.fujita060@gmail.com

Abstract

Philosophy systematically investigates fundamental questions concerning reality, knowledge, values, mind, language, logic, meaning, methodology, and reasoning. It emphasizes evidence, argument, clarity, and wisdom. The term *Structure* in this paper refers to any kind of construction, whether in mathematical or real-life contexts. In other words, it includes structures arising in science, engineering, mathematics, biology, chemistry, medicine, social science, physics, education, and psychology. Since both Philosophy and Structure can be regarded as origins of concepts in any domain, their study provides approaches that can be applied to any concept in any field. Nevertheless, the perspectives known through Philosophy and Structure cannot yet be considered universally recognized. To bridge this gap, the present paper offers reflections on various forms of Philosophy and on various forms of Structure.

Keywords: Philosophy, Structure, MetaPhilosophy, HyperPhilosophy

1 Introduction

1.1 Many Types of Philosophy

Philosophy systematically investigates fundamental questions about reality, knowledge, value, mind, language, logic, meaning, methodology, and reasoning, emphasizing evidence, argument, clarity, and wisdom (cf. [1–4]). Philosophy serves as the foundation of all scientific research—spanning science, engineering, mathematics, biology, chemistry, medicine, social science, physics, education, and psychology—and the concepts developed within philosophy have found applications across these disciplines. An overview of the conceptual domains within Philosophy is presented in Table 1. Moreover, various extensions of Philosophy are known.

Table 1: Conceptual domains within Philosophy

No.	Domain	Focus
1	Metaphysics	Nature of reality and being
2	Epistemology	Theory of knowledge
3	Logic	Principles of valid reasoning
4	Ethics	Moral principles and value judgments
5	Aesthetics	Nature of beauty and art
6	Political Philosophy	Justice, authority, rights
7	Philosophy of Language	Meaning, reference, communication
8	Philosophy of Mind	Consciousness, cognition, mental states
9	Philosophy of Science	Methods, foundations of sciences
10	Philosophy of Mathematics	Foundations, abstraction, truth in math
11	Philosophy of Physics	Space, time, causality, theory interpretation
12	Philosophy of Biology	Life, evolution, reductionism vs. holism
13	Philosophy of Technology	Tools, artifacts, human–machine relations
14	Philosophy of Religion	Faith, existence of God, metaphysical theology
15	Philosophy of Education	Aims, methods, values in learning

1.2 Many Types of Mathematical/Real-Life Structure

The term *Structure* here refers to any kind of construction in mathematical or real-life contexts. In other words, it encompasses structures in science, engineering, mathematics, biology, chemistry, medicine, social science, physics, education, and psychology. Table 2 presents an overview of various mathematical structures. Here, the term *structure* broadly denotes fundamental formal systems used to model diverse phenomena.

Table 2: Overview of Structure Types

Structure	Description
Set	A carrier with distinguished elements or relations but no additional operations.
Logic	$(L, \wedge, \vee, \neg)$ with connectives satisfying logical axioms.
Probability space [5]	$(\Omega, \mathcal{F}, P)$ with P a probability measure on $\mathcal{F}$.
Statistical model [6]	$(X, \mathcal{A}, \theta)$ mapping data X to statistical parameters.
Group [7]	$(G, *)$ with associative law, identity, and inverses.
Ring [8]	$(R, +, \times)$ satisfying ring axioms.
Vector space [9]	$(V, +, \cdot)$ over a field $\mathbb{F}$ with addition and scalar multiplication.
Metric space [10]	(X, dist) with distance satisfying non-negativity, symmetry, triangle inequality.
Graph [11]	(V, E) with edges E connecting vertex pairs.
Automaton [12]	$(Q, \Sigma, \delta, q_0, F)$ finite-state machine with transitions and accept states.
Game [13]	$(N, \{A_i\}, \{u_i\})$ with players N , actions A_i , and utilities u_i .

Table 3: XXX-Structure (Overview)

Name	Objects & operations	Essence
Classical	$H; \#^{(m)} : H^m \rightarrow H$ (relations optional)	Usual algebra/relational setting.
MetaStructure	$U \subseteq \text{Str}; \Phi : U^k \rightarrow U$ (iso-invariant)	Constructors on structures.
Iterated Meta	Levels $U^{(i)}$ with lifts $\Phi^\dagger$	Meta-hierarchy.
Hyper	$\mathcal{P}(S); \circ : \mathcal{P}(S)^2 \rightarrow \mathcal{P}(S)$	Set-valued ops.
SuperHyper (m, n)	$\mathcal{P}^m(S); \odot^{(m,n)} : (\mathcal{P}^m)^s \rightarrow \mathcal{P}^n$	Cross-level ops on nested sets.
Multi	$H; \#^{(m)} : H^m \rightarrow \mathcal{M}(H)$	Outputs with multiplicity.
Iterative Multi (k)	$\mathcal{M}^i(H); \#^{(m,i)} : (\mathcal{M}^i)^m \rightarrow \mathcal{M}^{i+1}$	Bags-of-bags.
Curried j -ary	$H; \text{curry } \#^{(m)} \text{ as } f^{(m)} : H \rightarrow \dots \rightarrow H$	Encode arity via unary maps.
TreeStructure	Rooted tree poset $(T, \leq, r); \star : T^m \rightarrow T$	Single-root hierarchy.
ForestStructure	Disjoint rooted trees $(F, \leq); \star : F^m \rightarrow F$	Multi-root collection.
Functorial	Functor $F : \mathcal{C} \rightarrow \text{Str}; F(f)$ transports	Functorial morphism transport.
U-Structure	$C_U = (H, \mu, \{\#^{(m)}\}); \mu : H \rightarrow U$	Ops with U -labels.
U-Off	$C_{U,\text{off}} = (H, \mu_{i,k}, \{\#^{(m)}\}); \mu : H \rightarrow]i, k[$	Allows values outside $[0, 1]$.
Rough	$C_R = (H, E, L_R, U_R, \{\#^{(m)}\}); E$ indiscernibility; lifted $\#_R^{(m)}$ via (L, U)	Rough-set extension.

Table 3 lists concepts recently proposed in research on structures. As seen there, any concept can be regarded as a Structure, and investigations have begun into the properties of such Structures.

Philosophy frames fundamental questions, clarifies concepts, and sets standards for reasoning. Structure supplies the formal scaffolds—sets, graphs, logics, models—used to represent and analyze phenomena. Together they coevolve: philosophy guides what should be modeled and why; Structure tests ideas, reveals patterns, and enables cross-disciplinary unification.

1.3 Our Contribution

From the above considerations, research on both Philosophy and Structures can be regarded as a fundamental basis for any concept in any scientific field, serving as an origin point of inquiry and thus of utmost importance. However, especially in the case of Structures, existing research cannot yet be considered sufficient; current theories and notions are not widely recognized, and applications across diverse domains remain limited.

To narrow this gap, the present paper provides reflections on Philosophy (Table 4) and on various forms of Structure (Table 5).

Several of the philosophical and structural notions discussed here are newly defined within this work. The main aim of this paper is to show that concepts in any domain can be extended through the approaches introduced here. In this paper, while we attempt to provide mathematically precise definitions as much as possible, it should be kept in mind that these definitions are only examples.

1.4 Contents in this paper

The remainder of this paper is organized as follows.

1	Introduction	1
1.1	Many Types of Philosophy	1
1.2	Many Types of Mathematical/Real-Life Structure	1
1.3	Our Contribution	2
1.4	Contents in this paper	2

Table 4: Concise overview of formal “Philosophies”

Name	Core objects / rules	Essence
Philosophy	$(Q, A, \text{adr}, \text{Cn})$ (Tarskian closure)	Map questions to justified answers.
Metaphilosophy	Meta-theory about $(Q, \text{Ans}, \text{Meth}, \text{Norm})$	Examine aims, methods, norms.
Iterated Metaphilosophy	Levels $\Phi^{(k+1)} = \uparrow(\Phi^{(k)})$	Multi-level self-evaluation.
HyperPhilosophy	Hyperop $\star : A \times A \rightarrow \mathcal{P}(A)$	Set-valued consequences.
SuperHyperPhilosophy	$\odot^{(m,n)}$ on $\mathcal{P}^k(A)$; flatten	Nested sets, then realize to base.
MultiPhilosophy	Multisets $\mathcal{M}(A)$; multi-ops $\diamond$	Track multiplicity/evidence.
Iterated MultiPhilosophy	Levels $\mathcal{M}^i(A)$; lifts; flatten	Bags-of-bags; collapse to base.
Fuzzy Philosophy	Degrees $\mu \in L^A$; fuzzy rules; t -norm T	Degree-valued inference.
Neutrosophy (Neutrosophic Philosophy)	$v(A, R) = (T, I, F) \subseteq]0, 1]^3$	Context-relative, independent ranges.
Plithogenic-Philosophy	(A, Q, V, D, adr) ; $c : V^2 \rightarrow [0, 1]$, v_D ; T, S	Mix T/S via $c_v = c(v_D, v)$; $\text{Agg}(x, y) = (1 - c_v)T(x, y) + c_v S$; subsumes fuzzy/intuitionistic/neutrosophic.
Functorial-Philosophy	Functor $\mathbf{P} : C \rightarrow \mathbf{PhilStr}$	Transport structures along morphisms.
Non-Philosophy	Invariants $\text{Inv}(\mathbf{P})$ of decisional split	“Science of philosophy” via invariants.
Anti-Philosophy	Partial semantics Sem ; domain-only closure	Dissolve ill-posed; reason operationally.
Critical Philosophy	$\mathcal{L}_{\text{crit}}$, Ax_{tr} on $\mathbb{E}, \mathbb{N}$	Limit knowledge to appearances/forms.
Dual-Philosophy	Contravariant D , $\varepsilon : \text{Id} \Rightarrow D^2$	Dualize structures.
Iterated Dual	Iterates D^n	Even $\cong$ original; odd $\cong$ dual.
Multipolar-Philosophy	Poles P with $v_P : A \rightarrow L$, weights w_P	Aggregate perspectives; no single pole.

Table 5: XXX-Structure in this paper (Overview)

Name	Objects & operations	Essence
Classical	$H; \#^{(m)} : H^m \rightarrow H$ (relations optional)	Usual algebra/relational setting.
MetaStructure	$U \subseteq \text{Str}_{\Sigma}; \Phi : U^k \rightarrow U$ (iso-invariant)	Constructors on structures.
Iterated Meta	Levels $U^{(i)}$ with lifts $\Phi^\uparrow$	Meta-hierarchy.
Hyper	$\mathcal{P}(S); \circ : \mathcal{P}(S)^2 \rightarrow \mathcal{P}(S)$	Set-valued ops.
SuperHyper (m, n)	$\mathcal{P}^m(S); \odot^{(m,n)} : (\mathcal{P}^m)^s \rightarrow \mathcal{P}^n$	Cross-level ops on nested sets.
Multi	$H; \#^{(m)} : H^m \rightarrow \mathcal{M}(H)$	Outputs with multiplicity.
Iterative Multi (k)	$\mathcal{M}^i(H); \#^{(m,i)} : (\mathcal{M}^i)^m \rightarrow \mathcal{M}^{i+1}$	Bags-of-bags.
Curried j -ary	H ; curry $\#^{(m)}$ as $f^{(m)} : H \rightarrow \dots \rightarrow H$	Encode arity via unary maps.
Functorial	Functor $F : C \rightarrow \mathbf{Str}$; $F(f)$ transports	Functorial morphism transport.

2 Preliminaries	4
2.1 Classical Structure	4
2.2 Philosophy	5
3 Metaphilosophy (Philosophy of Philosophy)	6
4 Iterated Metaphilosophy (Philosophy of ... of Philosophy)	7
5 HyperStructure and HyperPhilosophy	8
6 SuperHyperStructure and SuperHyperPhilosophy	10
7 MultiStructure and MultiPhilosophy	12
8 Iterated MultiStructure and Iterated MultiPhilosophy	14
9 Fuzzy Set and Fuzzy Philosophy	16
10 Neutrosophic Set and Neutrosophy	18
11 Plithogenic Set and Plithogeny (Plithogenic-Philosophy)	20
12 Functorial Structure and Functorial-Philosophy	22
13 Anti-Structure and Anti-Philosophy	25
14 Non-Philosophy	27
15 Critical Philosophy	28
16 Partial Philosophy	29
17 Dual-Structure and Dual-philosophy	29
18 Iterated Dual-Structure and Iterated Dual-philosophy	31
19 Multipolar-Structure and Multipolar-Philosophy	32
20 Pseudophilosophy	33
21 Many Other Structures	34

22 Many Other Philosophy	37
23 Combined Thought of Structure/Philosophy	39
24 Conclusion	41

2 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Throughout this paper, all structures and sets are assumed to be finite.

2.1 Classical Structure

In this paper, the term *Structure* is understood broadly as a mathematical system, not confined to a single discipline, but encompassing domains such as Set Theory, Logic, Probability, Statistics, Algebra, and Geometry. Moreover, the notion of Structure here is not limited to mathematics alone, but also refers to concepts and constructions arising in any aspect of real life.

Definition 2.1 (Classical Structure). (cf. [14, 15]) A *Classical Structure* C is a mathematical object arising from a traditional field—for example Set Theory, Logic, Probability, Statistics, Algebra, Geometry, Graph Theory, Automata Theory, or Game Theory. Formally, it may be represented as a pair

$$C = (H, \{\#^{(m)}\}_{m \in I}),$$

where:

- H is a nonempty set, often called the *carrier* or *universe*.
- For each $m \in I \subseteq \mathbb{Z}_{>0}$, there exists an m -ary operation

$$\#^{(m)} : H^m \longrightarrow H,$$

subject to appropriate *axioms* (such as associativity, commutativity, or identity laws), which vary according to the chosen type of structure.

The collection $\{\#^{(m)} : m \in I\}$ determines the *type* of C . Representative examples include:

- A *Set* $(S, \emptyset)$, consisting solely of a carrier with distinguished elements or relations, but without operations [16, 17].
- A *Logic* structure $(L, \wedge, \vee, \neg)$, where $\wedge, \vee$ are binary connectives and $\neg$ a unary connective, satisfying logical axioms [18].
- A *Probability* model $(\Omega, \mathcal{F}, P)$, where $P : \mathcal{F} \rightarrow [0, 1]$ is a probability measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ [19, 20].
- A *Statistical* model $(X, \mathcal{A}, \theta)$, where θ maps data X into parameters of interest [21, 22].
- *Algebraic* structures such as:
 - A *Group* $(G, *)$, with $* : G \times G \rightarrow G$ satisfying associativity, identity, and inverses [23, 24].
 - A *Ring* $(R, +, \times)$, with two binary operations fulfilling ring axioms [8, 25].
 - A *Vector Space* $(V, +, \cdot)$ over a field $\mathbb{F}$, with scalar multiplication $\cdot : \mathbb{F} \times V \rightarrow V$ [26–28].
- A *Geometric* structure (X, dist) , where $\text{dist} : X \times X \rightarrow \mathbb{R}$ satisfies the axioms of a metric.
- A *Graph* (V, E) , where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ for undirected graphs, or $E \subseteq V \times V$ for directed graphs, with adjacency and incidence relations [29–32].
- An *Automaton* $(Q, \Sigma, \delta, q_0, F)$, where Q is a set of states, Σ an input alphabet, $\delta : Q \times \Sigma \rightarrow Q$ the transition function, $q_0 \in Q$ the start state, and $F \subseteq Q$ the accepting states [33, 34].
- A *Game* $(N, \{A_i\}, \{u_i\})$, where N is the set of players, A_i each player's action set, and $u_i : \prod_{j \in N} A_j \rightarrow \mathbb{R}$ the payoff function for player i [35, 36].

Related concepts include the *HyperStructure* [37–40] and the *SuperHyperStructure* [41–44], which have also been extensively investigated in recent studies.

2.2 Philosophy

Philosophy systematically investigates fundamental questions about reality, knowledge, value, mind, language, logic, meaning, methodology, and reasoning. evidence, argument, clarity, wisdom (cf. [1–4]).

Definition 2.2 (Philosophy (as a formal system of inquiry)). Let Q be a nonempty set of *questions* and A a nonempty set of *propositions* equipped with an *addressing map*

$$\text{adr} : A \longrightarrow Q \quad (\text{adr}(a) \text{ is the question that } a \text{ addresses}).$$

A *philosophy* is a triple

$$\mathcal{P} = (Q, A, \text{Cn}),$$

where $\text{Cn} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is a Tarskian consequence operator (extensive, monotone, idempotent). For each $q \in Q$, the set of *acceptable answers* delivered by $\mathcal{P}$ is

$$\text{Ans}_{\mathcal{P}}(q) := \bigcup_{\Gamma \subseteq A} \left(\text{Cn}(\Gamma) \cap \text{adr}^{-1}(\{q\}) \right).$$

Intuitively, a philosophy systematically maps questions to (justified) answers by means of an admissible consequence relation. Optional *methods* and *norms* may be folded into Cn (e.g. by restricting which Γ are permitted), but are not required in this minimal definition.

Example 2.3 (Conceptual domains within Philosophy). The following 15 major conceptual domains are commonly investigated under the umbrella of Philosophy:

1. Metaphysics (nature of reality and being) [45, 46],
2. Epistemology (theory of knowledge) [47, 48],
3. Logic (principles of valid reasoning),
4. Ethics (moral principles and value judgments),
5. Aesthetics (nature of beauty and art),
6. Political Philosophy (justice, authority, rights) [49, 50],
7. Philosophy of Language (meaning, reference, communication) [51],
8. Philosophy of Mind (consciousness, cognition, mental states) [52],
9. Philosophy of Science (methods, foundations of sciences) [53, 54],
10. Philosophy of Mathematics (foundations, abstraction, truth in math) [55, 56],
11. Philosophy of Physics (space, time, causality, interpretation of theories) [57, 58],
12. Philosophy of Biology (life, evolution, reductionism vs. holism) [59, 60],
13. Philosophy of Technology (tools, artifacts, human–machine relations) [61, 62],
14. Philosophy of Religion (faith, existence of God, metaphysical theology) [63, 64],
15. Philosophy of Education (aims, methods, values in learning).

Each of these domains may be modeled as a restriction of the formal system $\mathcal{P} = (Q, A, \text{Cn})$ in Definition 2.2, where Q and A are drawn from the specific conceptual field under study.

Example 2.4 (Commute route selection). Let $Q = \{q = \text{“commute choice?”}\}$ and

$A = \{r = \text{“rain”}, t = \text{“heavy traffic”}, s = \text{“sunny”}, \ell = \text{“low traffic”}, h = \text{“take highway”}, m = \text{“take metro”}\},$

with $\text{adr}(h) = \text{adr}(m) = q$. Assume Cn is the least consequence operator closing under rules

$$\mathcal{R} = \{r \Rightarrow m, t \Rightarrow m, s \wedge \ell \Rightarrow h\}.$$

For premises $\Gamma = \{r\}$ we get $\text{Cn}(\Gamma) \supseteq \{m\}$, hence

$$\text{Ans}_{\mathcal{P}}(q) = \{m\}.$$

Example 2.5 (Dinner planning at home). Let $Q = \{q = \text{"dinner plan?"}\}$ and

$A = \{e = \text{"empty fridge"}, d = \text{"pasta on discount"}, nd = \text{"no discount"}, c = \text{"cook pasta"}, o = \text{"eat out"}\}$,
with $\text{adr}(c) = \text{adr}(o) = q$. Let Cn close under

$$\mathcal{R} = \{e \wedge d \Rightarrow c, e \wedge nd \Rightarrow o\}.$$

For $\Gamma = \{e, d\}$ we have $\text{Cn}(\Gamma) \supseteq \{c\}$, so

$$\text{Ans}_{\mathcal{P}}(q) = \{c\}.$$

Example 2.6 (Clinic triage). Let $Q = \{q = \text{"next action?"}\}$ and

$A = \{f = \text{"fever"}, g = \text{"cough"}, b = \text{"short breath"}, t = \text{"test for influenza"}, i = \text{"isolate"}, u = \text{"urgent care"}\}$,
with $\text{adr}(t) = \text{adr}(i) = \text{adr}(u) = q$. Let Cn close under

$$\mathcal{R} = \{f \wedge g \Rightarrow t, g \wedge b \Rightarrow u, g \wedge b \Rightarrow i\}.$$

With $\Gamma = \{f, g\}$ we infer $t \in \text{Cn}(\Gamma)$, hence

$$\text{Ans}_{\mathcal{P}}(q) = \{t\}.$$

Example 2.7 (Monthly personal finance action). Let $Q = \{q = \text{"action this month?"}\}$ and

$A = \{s = \text{"budget surplus"}, h = \text{"high-interest debt"}, nh = \text{"no high-interest debt"},$

$p = \text{"pay down debt"}, v = \text{"invest in index fund"}\}$,

with $\text{adr}(p) = \text{adr}(v) = q$. Let Cn close under

$$\mathcal{R} = \{s \wedge h \Rightarrow p, s \wedge nh \Rightarrow v\}.$$

For $\Gamma = \{s, h\}$ we get $\text{Cn}(\Gamma) \supseteq \{p\}$, so

$$\text{Ans}_{\mathcal{P}}(q) = \{p\}.$$

Although this reiterates part of the introduction, Philosophy frames fundamental questions, clarifies concepts, and sets standards for reasoning. Structure provides the formal scaffolds—sets, graphs, logics, models—used to represent and analyze phenomena. Together they coevolve: philosophy guides what should be modeled and why; Structure tests ideas, reveals patterns, and enables cross-disciplinary unification.

3 Metaphilosophy (Philosophy of Philosophy)

A MetaStructure is a higher-order framework where operations act on entire structures, producing new structures while preserving isomorphisms(cf. [65–68]). Metaphilosophy studies philosophy itself: its aims, methods, questions, boundaries, progress, standards of evaluation, practices, institutions, pedagogy, communication, history, value, and limitations (cf. [69–72]).

Definition 3.1 (Signature and Σ -structures). (cf. [65–68]) A (single-sorted, finitary) *signature* is a tuple

$$\Sigma = (\text{Func}, \text{Rel}, \text{ar}_{\text{Func}}, \text{ar}_{\text{Rel}}),$$

where Func and Rel are sets of function and relation symbols, and ar gives their arities. A Σ -*structure* is a tuple

$$\mathbf{C} = (H, (f^{\mathbf{C}})_{f \in \text{Func}}, (R^{\mathbf{C}})_{R \in \text{Rel}}),$$

with nonempty carrier H , each $f^{\mathbf{C}} : H^m \rightarrow H$ for $m = \text{ar}(f)$, and each $R^{\mathbf{C}} \subseteq H^r$ for $r = \text{ar}(R)$. Let Str_{Σ} be the class of all Σ -structures.

Definition 3.2 (MetaStructure over a fixed signature). (cf. [68]) Given Σ as in Theorem 3.1, a *MetaStructure* is a pair

$$\mathbb{M} = (U, (\Phi_{\ell})_{\ell \in \Lambda}),$$

where:

- $U \subseteq \text{Str}_\Sigma$ is a nonempty collection of Σ -structures;
- each *meta-operation* $\Phi_\ell : U^{k_\ell} \rightarrow U$ (meta-arity $k_\ell \in \mathbb{N}$) is specified by uniform constructors on carriers and symbol-interpretations.

Isomorphism invariance (naturality): if $\alpha_i : \mathbf{C}_i \cong \mathbf{D}_i$ in Str_Σ , then $\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}) : \Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) \cong \Phi_\ell(\mathbf{D}_1, \dots, \mathbf{D}_{k_\ell})$.

Definition 3.3 (Metaphilosophy). A (first-order) *philosophical theory* is a quadruple

$$\Phi = (\mathbf{Q}, \text{Ans}, \text{Meth}, \text{Norm}),$$

where $\mathbf{Q}$ is a set of questions, Ans candidate answers, Meth admissible methods, and Norm evaluation norms (e.g. validity, clarity, fruitfulness). A *metaphilosophical theory* $\Phi^\uparrow$ is a theory in a meta-language whose nonlogical symbols *range over* the components of Φ and whose axioms assert properties and constraints about them, e.g.

$$\text{WellFormed}(\mathbf{q}), \quad \text{Admissible}(\mathbf{m}), \quad \text{Adequate}(\mathbf{n}, \mathbf{q}, \mathbf{a}).$$

In short, $\Phi^\uparrow$ studies and regulates the aims, methods, and norms of Φ .

Example 3.4 (Departmental policy (object vs. meta level)).

- **Object level** $\Phi^{(0)}$. Faculty investigate questions (e.g. moral realism, modality) using methods such as argument analysis, formal modeling, experiments, or history of ideas, and publish answers.
- **Meta level** $\Phi^{(1)} \models \Phi^{(0)}$. A committee debates which questions are central ($\mathbf{Q}$), which methods are admissible (Meth), and which norms guide evaluation (Norm : clarity, reproducibility, impact). The resulting rules and policies constitute a concrete metaphilosophical theory about $\Phi^{(0)}$.

4 Iterated Metaphilosophy (Philosophy of ... of Philosophy)

An *Iterated MetaStructure* is obtained by repeatedly applying the MetaStructure construction, thereby forming successive levels where “structures of structures” build a hierarchical tower (cf. [65–68]). Iterated metaphilosophy recursively studies higher-order layers: metaphilosophies evaluating other metaphilosophies, formalizing reflexive methods, coherence, convergence, governance, incentive structures. dynamics, impact.

Definition 4.1 (Iterated MetaStructure of depth t). (cf. [68]) For $t \in \mathbb{N}$, an *Iterated MetaStructure of depth t* over Σ is a MetaStructure $\mathfrak{M}^{(t)}$ obtained by t iterations of a lifting procedure. When $s < t$, we *lift* a height- s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{S_j\})$ to height t by

$$\iota_{s \rightarrow t} : U^{(s)} \xrightarrow{U_\Sigma^{t-s}} U^{(t)} := U_\Sigma^{t-s}(U^{(s)}),$$

and, for each meta-operation $\odot_i : (E_\Sigma^{m_i})^{k_i} \rightarrow \mathcal{P}^{n_i}(E_\Sigma^{n_i})$, define its lift by

$$\odot_i^\uparrow : (E_\Sigma^{m_i+t-s})^{k_i} \longrightarrow \mathcal{P}^{n_i}(E_\Sigma^{n_i+t-s}), \quad \odot_i^\uparrow(U_\Sigma^{t-s}(x_1), \dots, U_\Sigma^{t-s}(x_{k_i})) := U_\Sigma^{t-s}(\odot_i(x_1, \dots, x_{k_i})),$$

and analogously for relations $S_j^\uparrow := (U_\Sigma^{t-s})^{\times \ell_j}(S_j)$.

Definition 4.2 (Iterated Metaphilosophy of depth t). Fix a base philosophical theory

$$\Phi^{(0)} = (\mathbf{Q}^{(0)}, \text{Ans}^{(0)}, \text{Meth}^{(0)}, \text{Norm}^{(0)})$$

. Define a meta-operator

$$\uparrow : \Phi^{(k)} \longmapsto \Phi^{(k+1)} := (\mathbf{Q}^{(k+1)}, \text{Ans}^{(k+1)}, \text{Meth}^{(k+1)}, \text{Norm}^{(k+1)}),$$

where each component of level $k+1$ is *about* level k :

$$\begin{aligned} \mathbf{Q}^{(k+1)} &:= \{ \text{questions about } (\mathbf{Q}^{(k)}, \text{Ans}^{(k)}, \text{Meth}^{(k)}, \text{Norm}^{(k)}) \}, \\ \text{Ans}^{(k+1)} &:= \{ \text{propositions/claims answering elements of } \mathbf{Q}^{(k+1)} \}, \\ \text{Meth}^{(k+1)} &:= \{ \text{meta-methods that evaluate/compare } \text{Meth}^{(k)} \text{ and } \text{Norm}^{(k)} \}, \\ \text{Norm}^{(k+1)} &:= \{ \text{norms to appraise elements of } \text{Ans}^{(k+1)} \text{ and } \text{Meth}^{(k+1)} \}. \end{aligned}$$

For $t \in \mathbb{N}$, the *iterated metaphilosophy tower of depth t* is

$$\Phi^{(\leq t)} := (\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(t)}), \quad \text{with } \Phi^{(k+1)} = \Uparrow(\Phi^{(k)}).$$

When needed, we write $\mathcal{L}^{(k)}$ for the (meta)-language in which $\Phi^{(k)}$ is formulated, so that $\mathcal{L}^{(k+1)}$ extends $\mathcal{L}^{(k)}$ by symbols that range over the components of $\Phi^{(k)}$.

Example 4.3 (Hospital medication safety). **Object** ($\Phi^{(0)}$): Clinicians follow a medication protocol P to reduce errors.

Meta ($\Phi^{(1)}$): The patient–safety committee audits P using KPIs (error rate, near misses) and updates P .

Meta² ($\Phi^{(2)}$): An external accreditor reviews the *audit method* (choice of KPIs, sampling, causality checks) and mandates improvements to the committee’s evaluation procedure.

Example 4.4 (Software engineering workflow). **Object** ($\Phi^{(0)}$): Developers ship code via feature branches and code review.

Meta ($\Phi^{(1)}$): The team runs retrospectives to assess the review process itself (latency, defect escape rate) and refines review checklists.

Meta² ($\Phi^{(2)}$): An org–level engineering excellence group evaluates *how* retrospectives are run (metrics validity, bias control) and standardizes evidence–based facilitation.

Example 4.5 (Academic peer review). **Object** ($\Phi^{(0)}$): Reviewers assess submitted papers.

Meta ($\Phi^{(1)}$): Area chairs meta–review reviews (calibrate scores, detect inconsistency) and issue guidance.

Meta² ($\Phi^{(2)}$): Program chairs/methods committee evaluate the meta–review rubric (inter–rater reliability, fairness metrics) and redesign the evaluation framework.

Example 4.6 (Public policy pilots). **Object** ($\Phi^{(0)}$): A city pilots a traffic policy and measures congestion.

Meta ($\Phi^{(1)}$): A policy unit evaluates the pilot’s *methodology* (counterfactuals, pre/post controls, survey design).

Meta² ($\Phi^{(2)}$): An oversight board audits the evaluators’ standards (statistical power, robustness, transparency) and codifies a meta–evaluation protocol for future pilots.

5 HyperStructure and HyperPhilosophy

A *HyperStructure* is a mathematical system $(\mathcal{P}(S), \circ)$ in which the operation $\circ : \mathcal{P}(S) \times \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ produces sets rather than single elements, thereby enabling the collective combination of subsets [41, 73–76]. Related notions include HyperGraph [77–80], HyperAlgebra [81–83], HyperUncertain Set [84–86], and HyperGame [13, 87, 88]. A *HyperPhilosophy* is a consequence framework

$$(A, Q, \text{adr}, M, R, \star)$$

in which arguments are combined through hyperoperations, producing sets of admissible answers that depend on both the chosen method and the surrounding context.

Definition 5.1 (Base set). [89, 90] Let S be a nonempty set, called the *base set*. Equivalently,

$$S = \{x \mid x \text{ is an element of the underlying universe}\}.$$

All further constructions—such as $\mathcal{P}(S)$ or its iterates $\mathcal{P}_n(S)$ —are formed from elements of S .

Definition 5.2 (Powerset). [90–92] The *powerset* of a set S , denoted $\mathcal{P}(S)$, is the collection of all subsets of S :

$$\mathcal{P}(S) = \{A \mid A \subseteq S\},$$

including both the empty set $\emptyset$ and S itself.

Definition 5.3 (Hyperoperation). (cf. [38, 39, 93]) A *hyperoperation* is a generalization of a binary operation in which the result of combining two inputs is a *set* (not necessarily a singleton). Formally, for a set S , a hyperoperation $\circ$ is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Definition 5.4 (Hyperstructure). (cf. [41, 73–76]) A *hyperstructure* replaces the base set S by its powerset and is given by

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where $\circ$ is a hyperoperation acting on subsets of S (i.e., $\circ : \mathcal{P}(S) \times \mathcal{P}(S) \rightarrow \mathcal{P}(S)$). This framework allows one to combine collections of elements into other collections.

Example 5.5 (Grocery bundle recommendation (Hyperstructure; real life)). Let

$$S = \{\text{pasta, rice, tomato, cheese, basil, oil}\}$$

be a catalog of items. For shopping lists $X, Y \subseteq S$, define a bundle-suggestion hyperoperation

$$X \circ Y := \begin{cases} (X \cup Y) \cup \{\text{basil, oil}\}, & \text{if } \{\text{pasta, tomato, cheese}\} \subseteq X \cup Y, \\ X \cup Y, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{P}(S), \circ)$ is a hyperstructure that maps two baskets to a *set* of recommended items. For instance, with $X = \{\text{pasta, tomato}\}$ and $Y = \{\text{cheese}\}$,

$$X \circ Y = \{\text{pasta, tomato, cheese, basil, oil}\},$$

modeling how combining two lists yields a (non-single) set of suggested additions.

Definition 5.6 (HyperPhilosophy (hyperstructural consequence system)). Let A be a nonempty set of *propositions* and Q a nonempty set of *questions*, with an *addressing map* $\text{adr} : A \rightarrow Q$ sending each proposition to the question it addresses. Let M be a nonempty set of *methods* (e.g., formal deduction, probabilistic inference, thought experiments), and R a nonempty set of *contexts* (e.g., background theories, empirical regimes).

A *HyperPhilosophy* is a tuple

$$\text{HP} := (A, Q, \text{adr}, M, R, \{ \star_{m,r} : m \in M, r \in R \}),$$

where for each $(m, r) \in M \times R$ the map

$$\star_{m,r} : A \times A \longrightarrow \mathcal{P}(A)$$

is a binary *hyperoperation of argumentative combination* (the set $\star_{m,r}(a, b)$ collects the admissible conclusions obtainable from jointly considering a and b by method m under context r).

For each fixed (m, r) we define the induced *hyperderivation operator*

$$\text{Der}_{m,r}^H(X) := X \cup \bigcup_{a,b \in X} \star_{m,r}(a, b), \quad X \subseteq A,$$

and the associated *hyperconsequence operator*

$$\text{Cn}_{m,r}^H(\Gamma) := \bigcup_{k=0}^{\infty} \Gamma_k, \quad \Gamma_0 := \Gamma, \quad \Gamma_{k+1} := \text{Der}_{m,r}^H(\Gamma_k).$$

For a question $q \in Q$, the set of *acceptable answers under (m, r) from premises Γ* is

$$\text{Ans}^H(q \mid \Gamma; m, r) := \text{Cn}_{m,r}^H(\Gamma) \cap \text{adr}^{-1}(\{q\}).$$

Example 5.7 (Emergency triage decisions (HyperPhilosophy; real life)). Let $A = \{a = \text{“fever”}, b = \text{“cough”}, c = \text{“shortness of breath”}, d = \text{“test for influenza”}, e = \text{“isolate patient”}, f = \text{“probable viral infection”}\}$, $Q = \{q = \text{“next action?”}\}$, and $\text{adr}(d) = \text{adr}(e) = \text{adr}(f) = q$. Take methods $M = \{m_{\text{guideline}}\}$ and contexts $R = \{r_{\text{winter}}\}$. Define the argumentative hyperoperation (set-valued inference) under $(m_{\text{guideline}}, r_{\text{winter}})$ by

$$\star_{m_{\text{guideline}}, r_{\text{winter}}}(a, b) = \{d, f\}, \quad \star_{m_{\text{guideline}}, r_{\text{winter}}}(b, c) = \{e\}.$$

From premises $\Gamma = \{a, b\}$ the one-step hyperderivation gives $\text{Der}_{m_{\text{guideline}}, r_{\text{winter}}}^H(\Gamma) = \Gamma \cup \{d, f\}$, so the acceptable answers are

$$\text{Ans}^H(q \mid \Gamma; m_{\text{guideline}}, r_{\text{winter}}) = \{d, f\},$$

illustrating method-and context-dependent *set-valued* conclusions in real triage.

Proposition 5.8 (Closure properties of $\text{Cn}_{m,r}^H$). *For each fixed $(m, r) \in M \times R$, the map $\text{Cn}_{m,r}^H : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is extensive, monotone, and idempotent, i.e., for all $\Gamma, \Delta \subseteq A$,*

$$(\text{Extensivity}) \quad \Gamma \subseteq \text{Cn}_{m,r}^H(\Gamma), \tag{1}$$

$$(\text{Monotonicity}) \quad \Gamma \subseteq \Delta \implies \text{Cn}_{m,r}^H(\Gamma) \subseteq \text{Cn}_{m,r}^H(\Delta), \tag{2}$$

$$(\text{Idempotence}) \quad \text{Cn}_{m,r}^H(\text{Cn}_{m,r}^H(\Gamma)) = \text{Cn}_{m,r}^H(\Gamma). \tag{3}$$

Proof. Fix (m, r) and abbreviate $\text{Der} := \text{Der}_{m,r}^H$, $\text{Cn} := \text{Cn}_{m,r}^H$. By definition, $\Gamma_0 = \Gamma$ and $\Gamma_{k+1} = \text{Der}(\Gamma_k) \supseteq \Gamma_k$, hence the union $\text{Cn}(\Gamma) = \bigcup_{k \geq 0} \Gamma_k$ contains Γ , proving (1).

If $\Gamma \subseteq \Delta$, then by induction on k one has $\Gamma_k \subseteq \Delta_k$: the base $k = 0$ is clear; and

$$\Gamma_{k+1} = \text{Der}(\Gamma_k) = \Gamma_k \cup \bigcup_{a,b \in \Gamma_k} \star_{m,r}(a, b) \subseteq \Delta_k \cup \bigcup_{a,b \in \Delta_k} \star_{m,r}(a, b) = \text{Der}(\Delta_k) = \Delta_{k+1}.$$

Taking unions over k gives $\text{Cn}(\Gamma) \subseteq \text{Cn}(\Delta)$, which is (2).

For idempotence, write $X := \text{Cn}(\Gamma) = \bigcup_{k \geq 0} \Gamma_k$. Then

$$\text{Der}(X) = X \cup \bigcup_{a,b \in X} \star_{m,r}(a, b) \supseteq \bigcup_{k \geq 0} \left(\Gamma_k \cup \bigcup_{a,b \in \Gamma_k} \star_{m,r}(a, b) \right) = \bigcup_{k \geq 0} \Gamma_{k+1} = X,$$

so the transfinite sequence $X, \text{Der}(X), \text{Der}^2(X), \dots$ stabilizes immediately at X . Hence $\text{Cn}(\text{Cn}(\Gamma)) = \bigcup_{j \geq 0} \text{Der}^j(X) = X = \text{Cn}(\Gamma)$. $\square$

6 SuperHyperStructure and SuperHyperPhilosophy

A *SuperHyperStructure* extends the concept of a hyperstructure by recursively applying the powerset construction n times [41]. In this framework, operations act on nested collections, thereby enabling the modeling of hierarchical, multi-level interactions [43, 94–96]. Related notions in the literature include the *SuperHyperGraph* [32, 97–99] and the *SuperHyperUncertain Set* [40, 100]. SuperHyperPhilosophy extends HyperPhilosophy by iterated powersets, enabling operations on nested collections to capture hierarchical multi-level reasoning.

Definition 6.1 (Iterated powersets). (cf. [41, 43, 83, 101]) For $n \geq 1$, define recursively

$$\mathcal{P}_1(S) = \mathcal{P}(S), \quad \mathcal{P}_{n+1}(S) = \mathcal{P}(\mathcal{P}_n(S)).$$

Similarly, the *nonempty* iterated powersets are

$$\mathcal{P}_1^*(S) = \mathcal{P}(S) \setminus \{\emptyset\}, \quad \mathcal{P}_{n+1}^*(S) = \mathcal{P}^*(\mathcal{P}_n^*(S)),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$.

Definition 6.2 (SuperHyperOperation). [41] Let H be a nonempty set. Define recursively, for each integer $k \geq 0$,

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)).$$

Fix $m, n \geq 0$. An (m, n) -*SuperHyperOperation* is a map

$$\odot^{(m,n)} : (\mathcal{P}^m(H))^s \longrightarrow \mathcal{P}^n(H)$$

for some input arity $s \in \mathbb{Z}_{>0}$. When the codomain is allowed to contain $\emptyset$, we speak of the *Neutrosophic* variant; otherwise it is the classical variant.

Definition 6.3 (SuperHyperStructure of order (m, n)). (cf. [42, 44, 93, 102]) Let S be a nonempty set and $m, n \geq 0$. A (m, n) -*SuperHyperStructure* on S with arity s is any choice of $\odot^{(m,n)} : (\mathcal{P}^m(S))^s \rightarrow \mathcal{P}^n(S)$. When $m = 0 = n$ one recovers ordinary s -ary operations on S ; when $m = 0, n = 1$ one recovers hyperoperations; when $s = 1$ one obtains *superhyperfunctions*.

Example 6.4 (Event playlist bundles (SuperHyperStructure; real life)). Let S be a finite set of songs. A *playlist* is a subset $X \in \mathcal{P}^1(S)$, and an *event program* is a set of playlists $Z \in \mathcal{P}^2(S)$. Define the $(m, n) = (1, 2)$ superhyperoperation with arity $s = 2$:

$$\odot^{(1,2)}(X, Y) := \{X, Y, X \cup Y, X \cap Y\} \setminus \{\emptyset\}, \quad X, Y \in \mathcal{P}^1(S).$$

Thus, given two playlists, $\odot^{(1,2)}$ returns a *set of playlists* collecting them, their mix ($X \cup Y$), and their overlap ($X \cap Y$) (if nonempty). For instance, with $X = \{s_1, s_2\}$ and $Y = \{s_2, s_3\}$,

$$\odot^{(1,2)}(X, Y) = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2\}\},$$

which is an element of $\mathcal{P}^2(S)$ (a set of playlists). This captures real-life “set-of-sets” recombination when curating event programs from existing playlists.

Definition 6.5 (SuperHyperPhilosophy (graded superhyper-consequence system)). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ the *addressing map* sending each proposition to the question it addresses. Let M be a nonempty set of *methods* and R a nonempty set of *contexts*. For each $(m, r) \in M \times R$ fix a family of labelled superhyperoperations

$$O_{m,r} := \left\{ \odot_{\ell}^{(m_{\ell}, n_{\ell})} : (\mathcal{P}^{m_{\ell}}(A))^{s_{\ell}} \rightarrow \mathcal{P}^{n_{\ell}}(A) \mid \ell \in \Lambda_{m,r} \right\},$$

with signatures $(s_{\ell}; m_{\ell}, n_{\ell})$ allowed to vary over ℓ .

Write $\mathbf{A}^{(k)} := \mathcal{P}^k(A)$ for the level- k domain. A *graded state* is a sequence $X = (X^{(k)})_{k \geq 0}$ with $X^{(k)} \subseteq \mathbf{A}^{(k)}$. Define the *one-step superhyper-derivation* at (m, r) by

$$\text{Der}_{m,r}^{\text{SH}}(X)^{(n)} := X^{(n)} \cup \bigcup_{\substack{\ell \in \Lambda_{m,r} \\ n_{\ell} = n}} \bigcup_{(x_1, \dots, x_{s_{\ell}}) \in (X^{(m_{\ell})})^{s_{\ell}}} \odot_{\ell}^{(m_{\ell}, n_{\ell})}(x_1, \dots, x_{s_{\ell}}).$$

Starting from initial premises $\Gamma \subseteq A$, set

$$X_0^{(0)} = \Gamma, \quad X_0^{(k)} = \emptyset \ (k \geq 1), \quad X_{t+1} = \text{Der}_{m,r}^{\text{SH}}(X_t) \quad (t \geq 0).$$

The *graded superhyper-consequence* of Γ is

$$\text{Cn}_{m,r}^{\text{SH}}(\Gamma)^{(n)} := \bigcup_{t \geq 0} X_t^{(n)} \quad (n \geq 0).$$

To read off base-level conclusions, define the *realization maps* $\text{Real}^{(k)} : \mathbf{A}^{(k)} \rightarrow \mathcal{P}(A)$ by

$$\text{Real}^{(0)}(a) = \{a\}, \quad \text{Real}^{(k+1)}(X) = \bigcup_{Y \in X} \text{Real}^{(k)}(Y).$$

The *flattened superhyper-consequence* on A is then

$$\text{Cn}_{m,r}^{0,\text{SH}}(\Gamma) := \bigcup_{n \geq 0} \bigcup_{x \in \text{Cn}_{m,r}^{\text{SH}}(\Gamma)^{(n)}} \text{Real}^{(n)}(x) \subseteq A.$$

For each question $q \in Q$, the set of *acceptable answers at (m, r) from Γ* is

$$\text{Ans}^{\text{SH}}(q \mid \Gamma; m, r) := \text{Cn}_{m,r}^{0,\text{SH}}(\Gamma) \cap \text{adr}^{-1}(\{q\}).$$

The tuple

$$\text{SHP} := (A, Q, \text{adr}, M, R, \{O_{m,r}\}_{(m,r) \in M \times R})$$

is called a *SuperHyperPhilosophy*.

Example 6.6 (Hiring pipeline with nested options (SuperHyperPhilosophy; real life)). Let $A = \{a = \text{“strong portfolio”}, b = \text{“good references”}, c = \text{“invite to onsite”}, d = \text{“assign take-home”}\}$ and $Q = \{q = \text{“next action?”}\}$ with $\text{adr}(c) = \text{adr}(d) = q$. Fix one method $M = \{m_{\text{policy}}\}$ and one context $R = \{r_{\text{normal}}\}$. Include two labelled superhyperoperations:

$$\begin{aligned} \odot_{\uparrow}^{(0,1)} : A \rightarrow \mathcal{P}^1(A), \quad \odot_{\uparrow}^{(0,1)}(a) &= \{c\}, \quad \odot_{\uparrow}^{(0,1)}(b) = \{d\}, \\ \odot_{\text{merge}}^{(1,1)} : \mathcal{P}^1(A) \times \mathcal{P}^1(A) &\rightarrow \mathcal{P}^1(A), \quad \odot_{\text{merge}}^{(1,1)}(X, Y) = X \cup Y. \end{aligned}$$

From premises $\Gamma = \{a, b\}$, the one-step graded update yields

$$X_1^{(1)} = \{\{c\}, \{d\}\}, \quad \text{hence} \quad \bigcup X_1^{(1)} = \{c, d\}.$$

Flattening to base level gives $\text{Cn}_{m_{\text{policy}}, r_{\text{normal}}}^{0,\text{SH}}(\Gamma) = \{c, d\}$, so

$$\text{Ans}^{\text{SH}}(q \mid \Gamma; m_{\text{policy}}, r_{\text{normal}}) = \{c, d\},$$

i.e., both “invite to onsite” and “assign take-home” are acceptable next actions generated via level-0 $\rightarrow$ 1 proposals and level-1 $\rightarrow$ 1 merging.

Example 6.7 (Weekend family planning (SuperHyperPhilosophy; real life)). Let

$$A = \{a = \text{“rainy”}, c = \text{“with kids”}, x = \text{“visit indoor museum”},$$

$$y = \text{“go to aquarium”}, w = \text{“join science workshop”}\},$$

$Q = \{q = \text{“plan options?”}\}$, and $\text{adr}(x) = \text{adr}(y) = \text{adr}(w) = q$. Fix $M = \{m_{\text{policy}}\}$, $R = \{r_{\text{weekend}}\}$. Define two labelled superhyperoperations:

$$\odot_{\uparrow}^{(0,1)} : A \rightarrow \mathcal{P}^1(A), \quad \odot_{\uparrow}^{(0,1)}(a) = \{x, y\}, \quad \odot_{\uparrow}^{(0,1)}(c) = \{w\},$$

$$\odot_{\text{bundle}}^{(1,2)} : \mathcal{P}^1(A) \times \mathcal{P}^1(A) \rightarrow \mathcal{P}^2(A), \quad \odot_{\text{bundle}}^{(1,2)}(X, Y) = \{X, Y, X \cup Y\} \setminus \{\emptyset\}.$$

From premises $\Gamma = \{a, c\}$,

$$X_1^{(1)} = \{\{x, y\}, \{w\}\}, \quad \odot_{\text{bundle}}^{(1,2)}(\{x, y\}, \{w\}) = \{\{x, y\}, \{w\}, \{x, y, w\}\} \subseteq X_2^{(2)}.$$

Flattening (realization) yields

$$\text{Cn}_{m_{\text{policy}}, r_{\text{weekend}}}^{0, \text{SH}}(\Gamma) = \{x, y, w\}.$$

Hence

$$\text{Ans}^{\text{SH}}(q \mid \Gamma; m_{\text{policy}}, r_{\text{weekend}}) = \{\text{visit indoor museum, go to aquarium, join science workshop}\}.$$

Proposition 6.8 (Tarskian properties). *For each fixed (m, r) , the graded operator $\text{Cn}_{m,r}^{\text{SH}}$ is extensive, monotone, and idempotent levelwise. Consequently, the flattened operator $\text{Cn}_{m,r}^{0, \text{SH}} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is also extensive, monotone, and idempotent.*

Proof. Fix (m, r) . Extensivity holds since $X_t \subseteq X_{t+1}$ by construction, hence $\Gamma = X_0^{(0)} \subseteq \bigcup_{t \geq 0} X_t^{(0)}$. Monotonicity: if $\Gamma \subseteq \Delta$, then $X_0 \subseteq Y_0$ and, inductively using the unions in $\text{Der}_{m,r}^{\text{SH}}$, $X_t \subseteq Y_t$ for all t , giving $\text{Cn}_{m,r}^{\text{SH}}(\Gamma) \subseteq \text{Cn}_{m,r}^{\text{SH}}(\Delta)$ levelwise. Idempotence: let $Z := \text{Cn}_{m,r}^{\text{SH}}(\Gamma)$. Since $\text{Der}_{m,r}^{\text{SH}}$ adds exactly those outputs already included in the definition of Z , one has $\text{Der}_{m,r}^{\text{SH}}(Z) = Z$, thus $\text{Cn}_{m,r}^{\text{SH}}(Z) = Z$. For the flattened operator, $\text{Real}^{(n)}$ is monotone in its argument and unions commute with unions, so extensivity/monotonicity/idempotence are preserved after flattening. $\square$

7 MultiStructure and MultiPhilosophy

A *MultiStructure* on a set H assigns to each m -tuple in H^m a finite multiset of elements drawn from $\mathcal{M}(H)$ [103, 104]. Related concepts include MultiGraph [105–107], Multialgebra [108, 109], MultiUncertain Set [110–112], and MultiTopology [113, 114]. MultiPhilosophy extends Philosophy by using multisets: arguments produce bag-valued conclusions, where multiplicities encode strength, redundancy, or evidence across methods and contexts.

Definition 7.1 (Finite multisets and basic operations). (cf. [115–118]) Let H be a nonempty set. The collection of *finite multisets* over H is

$$\mathcal{M}(H) := \{\mu : H \rightarrow \mathbb{N} \mid \#\text{supp}(\mu) < \infty\}, \quad \text{supp}(\mu) := \{x \in H : \mu(x) > 0\}.$$

For $\mu, \nu \in \mathcal{M}(H)$ and $k \in \mathbb{N}$ define, pointwise on H ,

$$(\mu \oplus \nu)(x) := \mu(x) + \nu(x), \quad (\mu \sqcup \nu)(x) := \max\{\mu(x), \nu(x)\}, \quad (k \odot \mu)(x) := k \cdot \mu(x).$$

Write $\mathbf{0} \in \mathcal{M}(H)$ for the zero multiset $\mathbf{0}(x) = 0$. We partially order $\mathcal{M}(H)$ by

$$\mu \preceq \nu \iff \forall x \in H (\mu(x) \leq \nu(x)).$$

Then $(\mathcal{M}(H), \preceq)$ is a join-semilattice with join $\sqcup$, and $(\mathcal{M}(H), \oplus, \mathbf{0})$ is a commutative monoid.

Definition 7.2 (MultiStructure). [103, 104] Let H be a nonempty set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ a set of arities. A *MultiStructure* on H is a pair

$$\mathcal{MS} = \left(H, \{ \#^{(m)} : H^m \rightarrow \mathcal{M}(H) \}_{m \in \mathcal{I}} \right).$$

For $\mathbf{a} = (a_1, \dots, a_m) \in H^m$, the multiset $\#^{(m)}(\mathbf{a})$ lists (with multiplicities) the outcomes produced from $\mathbf{a}$.

Example 7.3 (Task batching in a print shop (MultiStructure; real life)). Let $H = \{\text{Draft, Proof, Print, Pack, Ship}\}$ and take arities $\mathcal{I} = \{1, 2\}$. Define the multi-operations $\#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ by

$$\begin{aligned} \#^{(1)}(\text{Draft}) &= \{\text{Proof} : 1\}, & \#^{(1)}(\text{Proof}) &= \{\text{Print} : 200\}, \\ \#^{(2)}(\text{Print, Pack}) &= \{\text{Ship} : 200\}, & \text{and } \#^{(m)}(\mathbf{a}) &= \mathbf{0} \text{ otherwise.} \end{aligned}$$

Interpretation: a single *Draft* yields one *Proof*; one *Proof* yields a print run of 200 copies; combining a printed batch with *Pack* yields 200 *Ship* actions. Here the multiplicities in $\mathcal{M}(H)$ record how many identical tasks are produced.

Definition 7.4 (MultiPhilosophy (bag-valued consequence system)). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ the *addressing map*. Let M be a nonempty set of *methods* and R a nonempty set of *contexts*. For each $(m, r) \in M \times R$ fix a family of *multi-operations*

$$\mathcal{O}_{m,r} := \left\{ \diamond_\ell^{(s)} : A^s \rightarrow \mathcal{M}(A) \mid \ell \in \Lambda_{m,r}, s \in \mathbb{Z}_{>0} \right\}.$$

Interpretation: given input tuple $\mathbf{a} \in A^s$, the multiset $\diamond_\ell^{(s)}(\mathbf{a})$ contains all admissible conclusions, with their *multiplicity* representing strength, redundancy, or evidence count.

A *state of information* is a multiset $B \in \mathcal{M}(A)$. Define the *one-step multiderivation* at (m, r) by

$$\text{Der}_{m,r}^M(B) := B \sqcup \bigsqcup_{\ell \in \Lambda_{m,r}} \bigsqcup_{\mathbf{a} \in \text{supp}(B)^{s_\ell}} \diamond_\ell^{(s_\ell)}(\mathbf{a}) \in \mathcal{M}(A),$$

where s_ℓ is the arity of $\diamond_\ell$ and $\sqcup$ denotes join (pointwise max) in the poset $(\mathcal{M}(A), \preceq)$.

Given premises $\Gamma \subseteq A$, let $B_0 \in \mathcal{M}(A)$ be their indicator multiset $B_0(a) = 1$ for $a \in \Gamma$ and 0 otherwise. Define the iteration

$$B_{t+1} := \text{Der}_{m,r}^M(B_t) \quad (t \geq 0), \quad \text{Cn}_{m,r}^M(\Gamma) := \bigsqcup_{t \geq 0} B_t \in \mathcal{M}(A).$$

For each question $q \in Q$, the (bag-valued) set of *acceptable answers* from Γ at (m, r) is

$$\text{Ans}^M(q \mid \Gamma; m, r) := \text{Cn}_{m,r}^M(\Gamma) \upharpoonright \text{adr}^{-1}(\{q\}),$$

i.e. the restriction of the resulting multiset to those $a \in A$ with $\text{adr}(a) = q$.

Example 7.5 (Choosing a dinner spot with vote strength (MultiPhilosophy; real life)). Let the set of propositions be

$$A = \{c = \text{“cheap”}, n = \text{“near”}, r = \text{“raining”}, b = \text{“go to Bistro”}, f = \text{“go to Mall Food Court”}\}.$$

Questions are $Q = \{q_{\text{ctx}} = \text{“context?”}, q_{\text{dec}} = \text{“where to dine?”}\}$ with $\text{adr}(c) = \text{adr}(n) = \text{adr}(r) = q_{\text{ctx}}$ and $\text{adr}(b) = \text{adr}(f) = q_{\text{dec}}$. Fix one method $M = \{m_\star\}$ and one context $R = \{r_\star\}$. Introduce two multi-operations (bag-valued rules):

$$\begin{aligned} \diamond^{(2)}(c, n) &= \{b : 2\} \quad (\text{“cheap \& near” strongly supports Bistro}), \\ \Delta^{(1)}(r) &= \{f : 1\} \quad (\text{“raining” mildly supports Food Court}). \end{aligned}$$

From premises $\Gamma = \{c, n, r\}$ the one-step multiderivation adds the conclusions

$$\diamond^{(2)}(c, n) \oplus \Delta^{(1)}(r) = \{b : 2, f : 1\}.$$

Restricting to the decision question gives

$$\text{Ans}^M(q_{\text{dec}} \mid \Gamma; m_\star, r_\star) = \{b : 2, f : 1\},$$

i.e. “go to Bistro” has multiplicity 2 (stronger aggregated support) and “go to Mall Food Court” has multiplicity 1.

Proposition 7.6 (Tarskian properties in the multiset order). *For each fixed $(m, r) \in M \times R$, the operator $\text{Cn}_{m,r}^M : \mathcal{P}(A) \rightarrow \mathcal{M}(A)$, ordered by $\preceq$, is:*

$$\text{extensive} \quad (\iota(\Gamma) \preceq \text{Cn}_{m,r}^M(\Gamma)), \quad \text{monotone} \quad (\Gamma \subseteq \Delta \Rightarrow \text{Cn}_{m,r}^M(\Gamma) \preceq \text{Cn}_{m,r}^M(\Delta)),$$

and idempotent in the sense that if $B = \text{Cn}_{m,r}^M(\Gamma)$ then $\text{Der}_{m,r}^M(B) = B$, hence $\bigsqcup_{t \geq 0} \text{Der}^t(B) = B$. Here $\iota(\Gamma)$ is the indicator multiset of Γ .

Proof. Extensivity: $\text{Der}_{m,r}^M(B) \succeq B$ by the outer join; hence $B_{t+1} \succeq B_t$ and $\iota(\Gamma) = B_0 \preceq \bigsqcup_{t \geq 0} B_t$.

Monotonicity: if $\Gamma \subseteq \Delta$ then $B_0 \preceq B'_0$; by induction on t , the defining joins over $\text{supp}(B_t)$ are included in those over $\text{supp}(B'_t)$, so $B_t \preceq B'_t$ and $\bigsqcup_t B_t \preceq \bigsqcup_t B'_t$.

Idempotence: let $B := \bigsqcup_{t \geq 0} B_t$. Because every term added by $\text{Der}_{m,r}^M$ already occurs in some B_{t+1} , one has $\text{Der}_{m,r}^M(B) \preceq B$ while also $B \preceq \text{Der}_{m,r}^M(B)$, hence equality. $\square$

8 Iterated MultiStructure and Iterated MultiPhilosophy

An *Iterative MultiStructure* builds upon this by iterating the multiset construction through successive levels, yielding operations [103, 104, 119]. Iterated MultiPhilosophy extends MultiPhilosophy by iterating multiset levels, allowing reasoning over bags-of-bags, capturing hierarchical multiplicity and graded evidence across multiple abstraction layers.

Definition 8.1 (Iterated multiset levels). [103, 104, 119] Let X be a set. Define the levels

$$\mathcal{M}^0(X) := X, \quad \mathcal{M}^{i+1}(X) := \mathcal{M}(\mathcal{M}^i(X)) \quad (i \geq 0),$$

so $\mathcal{M}^1(X) = \mathcal{M}(X)$ is the collection of finite multisets on X , $\mathcal{M}^2(X)$ is the collection of finite multisets on $\mathcal{M}(X)$, etc. For $y \in \mathcal{M}^i(X)$ we denote by

$$\delta_y^{(i)} \in \mathcal{M}^{i+1}(X)$$

the *unit multiset at level $i+1$* that assigns multiplicity 1 to y and 0 to all other elements of $\mathcal{M}^i(X)$.

Definition 8.2 (Iterative Multi-Structure of order k). [103, 104, 119] Let H be a nonempty set, fix $k \in \mathbb{Z}_{\geq 1}$, and let $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ be a set of arities. An *Iterative Multi-Structure of order k* consists of a family of *level-wise multi-operations*

$$\text{IMS}^{(k)} := \left(H, \left\{ \#^{(m,i)} : (\mathcal{M}^i(H))^m \longrightarrow \mathcal{M}^{i+1}(H) \right\}_{m \in \mathcal{I}, 0 \leq i < k} \right).$$

For inputs $x_1, \dots, x_m \in \mathcal{M}^i(H)$, the value $\#^{(m,i)}(x_1, \dots, x_m)$ is an *output object* at the next level $\mathcal{M}^{i+1}(H)$.

Example 8.3 (Warehouse packing (Iterative Multi-Structure; real life)). Let $k = 2$ and $H = \{A, B\}$ denote two SKUs: $A = \text{“apple”}$, $B = \text{“banana”}$. Level 0 are single items (H); level 1 are *cartons* (finite multisets of items, $\mathcal{M}^1(H)$); level 2 are *cases of cartons* (finite multisets of cartons, $\mathcal{M}^2(H)$).

Define the level-wise multi-operations:

$$\#^{(1,0)}(A) = \{A : 6\}, \quad \#^{(1,0)}(B) = \{B : 4\}$$

(pack one apple-carton as 6 apples; one banana-carton as 4 bananas), and

$$\#^{(2,1)}(X, Y) = \{X : 1, Y : 1\} \quad \text{for } X, Y \in \mathcal{M}^1(H)$$

(build a *case* containing two cartons).

Trace. From items A and B we get cartons $X := \#^{(1,0)}(A) = \{A : 6\}$ and $Y := \#^{(1,0)}(B) = \{B : 4\}$. Bundling them yields a level-2 object

$$\#^{(2,1)}(X, Y) = \{X : 1, Y : 1\} \in \mathcal{M}^2(H),$$

i.e., one case with one apple-carton and one banana-carton.

Definition 8.4 (Iterated MultiPhilosophy of depth k). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ the *addressing map*. Let M be a nonempty set of *methods* and R a nonempty set of *contexts*. Fix $k \in \mathbb{Z}_{\geq 1}$.

For each $(m, r) \in M \times R$ and each level $i \in \{0, 1, \dots, k-1\}$, let

$$O_{m,r}^{(i)} := \left\{ \diamond_{\ell}^{(s,i)} : (\mathcal{M}^i(A))^s \longrightarrow \mathcal{M}^{i+1}(A) \mid \ell \in \Lambda_{m,r}^{(i)}, s \in \mathbb{Z}_{>0} \right\}$$

be a (possibly empty) family of level- i multi-operations that produce outputs at level $i+1$. A *graded state of information* is a tuple

$$\mathbf{B} = (B^{(0)}, B^{(1)}, \dots, B^{(k)}) \quad \text{with} \quad B^{(i)} \in \mathcal{M}^{i+1}(A) \quad (0 \leq i \leq k).$$

(Thus $B^{(0)} \in \mathcal{M}(A)$ is a multiset of base-level propositions; $B^{(1)}$ is a multiset of multisets of A ; etc.)

One-step graded derivation. Given (m, r) and a graded state $\mathbf{B}$, define $\text{Der}_{m,r}^{\text{IM}}(\mathbf{B})$ levelwise by

$$\text{Der}_{m,r}^{\text{IM}}(\mathbf{B})^{(0)} := B^{(0)},$$

and, for $i = 0, 1, \dots, k-1$,

$$\text{Der}_{m,r}^{\text{IM}}(\mathbf{B})^{(i+1)} := B^{(i+1)} \sqcup \bigsqcup_{\ell \in \Lambda_{m,r}^{(i)}} \bigsqcup_{(x_1, \dots, x_{s_{\ell}}) \in \text{supp}(B^{(i)})^{s_{\ell}}} \delta_{\diamond_{\ell}^{(s_{\ell},i)}(x_1, \dots, x_{s_{\ell}})}^{(i+1)}.$$

In words: each available input tuple at level i (picked from the support of $B^{(i)}$) is transformed by some operation $\diamond_{\ell}^{(s,i)}$ into a single output object at level $i+1$, which is then *inserted with multiplicity 1* into $B^{(i+1)}$ via the unit multiset $\delta_{(\cdot)}^{(i+1)}$; existing multiplicities are preserved by the join $\sqcup$.

Iteration from premises. Given premises $\Gamma \subseteq A$, set the initial graded state

$$B_0^{(0)}(a) = \begin{cases} 1, & a \in \Gamma, \\ 0, & a \notin \Gamma, \end{cases} \quad B_0^{(i)} = \mathbf{0} \quad (1 \leq i \leq k),$$

and define the sequence

$$\mathbf{B}_{t+1} := \text{Der}_{m,r}^{\text{IM}}(\mathbf{B}_t) \quad (t \geq 0).$$

The *graded (levelwise) closure* is

$$\text{Cn}_{m,r}^{\text{IM}}(\Gamma)^{(i)} := \bigsqcup_{t \geq 0} B_t^{(i)} \in \mathcal{M}^{i+1}(A) \quad (0 \leq i \leq k).$$

Flattening to base level. Define recursively the *realization maps*

$$\text{Real}^{[0]} : \mathcal{M}^1(A) \longrightarrow \mathcal{M}(A), \quad \text{Real}^{[0]}(\mu) := \mu,$$

$$\text{Real}^{[i+1]} : \mathcal{M}^{i+2}(A) \longrightarrow \mathcal{M}(A), \quad \text{Real}^{[i+1]}(\mu) := \bigoplus_{x \in \text{supp}(\mu)} \mu(x) \odot \text{Real}^{[i]}(x) \quad (i \geq 0).$$

The *flattened iterated multiconsequence* is then

$$\text{Cn}_{m,r}^{0,\text{IM}}(\Gamma) := \bigoplus_{i=0}^k \text{Real}^{[i]}(\text{Cn}_{m,r}^{\text{IM}}(\Gamma)^{(i)}) \in \mathcal{M}(A).$$

Finally, for $q \in Q$, the multiset of *acceptable answers* is the restriction

$$\text{Ans}^{\text{IM}}(q \mid \Gamma; m, r) := \text{Cn}_{m,r}^{0,\text{IM}}(\Gamma) \upharpoonright \text{adr}^{-1}(\{q\}).$$

The tuple

$$(A, Q, \text{adr}, M, R, \{O_{m,r}^{(i)}\}_{(m,r), 0 \leq i < k})$$

is called an *Iterated MultiPhilosophy of depth k* .

Example 8.5 (Two-stage hiring decision (Iterated MultiPhilosophy; real life)). Let the base propositions be

$$A = \{p = \text{“prepared”}, s = \text{“strong skills”}, h = \text{“hire”}, w = \text{“waitlist”}\}.$$

Questions are $Q = \{q_{\text{ctx}}, q_{\text{dec}}\}$ with $\text{adr}(p) = \text{adr}(s) = q_{\text{ctx}}$ and $\text{adr}(h) = \text{adr}(w) = q_{\text{dec}}$. Take depth $k = 2$ (panel level and board level), one method $M = \{m_\star\}$ and one context $R = \{r_\star\}$.

Level 0 → 1 (panel aggregation). Use a bag-valued rule:

$$\diamond^{(2,0)}(p, s) = \{h : 2\},$$

meaning “prepared & strong skills” yields strong (multiplicity 2) support to *hire*.

Level 1 → 2 (board docket). Wrap any panel bag $\mu \in \mathcal{M}^1(A)$ into a single report:

$$\diamond^{(1,1)}(\mu) = \{\mu : 1\} \in \mathcal{M}^2(A).$$

Trace from premises. With premises $\Gamma = \{p, s\}$, the level-1 multiset after one step is $\{h : 2\}$. The level-2 step produces a docket $\{\{h : 2\} : 1\}$. Flattening back to the decision level returns the bag $\{h : 2\}$, i.e., “hire” has stronger aggregated support than any alternative.

Proposition 8.6 (Tarskian properties). *Fix (m, r) . For each level i , the operator $\Gamma \mapsto \text{Cn}_{m,r}^{\text{IM}}(\Gamma)^{(i)}$ is extensive, monotone, and idempotent in the multiset order $\preceq$. Moreover, the flattened operator $\Gamma \mapsto \text{Cn}_{m,r}^{0,\text{IM}}(\Gamma)$ is extensive, monotone, and satisfies closure idempotence in the sense that, if $\mathbf{B} = \text{Cn}_{m,r}^{\text{IM}}(\Gamma)$, then*

$$\text{Der}_{m,r}^{\text{IM}}(\mathbf{B}) = \mathbf{B} \implies \text{Cn}_{m,r}^{0,\text{IM}}(\Gamma) = \bigoplus_{i=0}^k \text{Real}^{[i]}(B^{(i)}).$$

Proof. Levelwise extensivity and monotonicity follow from the outer join $\sqcup$ and the fact that supports only grow with t and with Γ . Idempotence holds because applying $\text{Der}_{m,r}^{\text{IM}}$ to the union $\bigsqcup_{t \geq 0} B_t^{(i)}$ produces no new outputs beyond those already present in some $B_{t+1}^{(i+1)}$. For flattening, $\text{Real}^{[i]}$ is monotone and $\oplus$ -additive by construction, hence the direct sum over levels preserves extensivity and monotonicity. If $\mathbf{B}$ is a fixed point of $\text{Der}_{m,r}^{\text{IM}}$, then its levelwise closures equal $B^{(i)}$, yielding the displayed identity. $\square$

9 Fuzzy Set and Fuzzy Philosophy

A *Fuzzy Set* assigns each element a membership degree in $[0, 1]$, representing graded belongingness rather than crisp inclusion [120–122]. Related concepts include the *Bipolar Fuzzy Set* [123, 124], *Hesitant Fuzzy Set* [125, 126], *Picture Fuzzy Set* [127, 128], *Spherical Fuzzy Set* [129, 130], and *Intuitionistic Fuzzy Set* [131, 132]. Moreover, applications of Fuzzy Sets and Fuzzy Structures have been studied in a wide variety of fields, including Control Theory [133–136], Neural Network [137–139], Decision-Making [140–142], Graph Theory [143, 144], and Algebra [145, 146]. A *Fuzzy Philosophy* models reasoning with graded truth values using t -norms and fuzzy rules, producing degree-valued answers rather than crisp true/false conclusions (cf. [147–150]).

Definition 9.1 (Fuzzy Set). [120, 151] Let Y be a nonempty domain. A *fuzzy set* is given by a function

$$\mu : Y \longrightarrow [0, 1],$$

where $\mu(y)$ measures the degree to which y belongs to the set. A *fuzzy relation* on Y is a function $\delta : Y \times Y \rightarrow [0, 1]$, viewed as a fuzzy subset of $Y \times Y$. We say δ is a *fuzzy relation on μ* if for every $y, z \in Y$,

$$\delta(y, z) \leq \min\{\mu(y), \mu(z)\}.$$

Definition 9.2 (Truth-degree algebra). Let $\mathbf{L} = (L, \leq, T, S, \Rightarrow, 0, 1)$ where $(L, \leq)$ is a complete lattice with bottom 0 and top 1, $T : S^2 \rightarrow L$ is a left-continuous t -norm, $S : L^2 \rightarrow L$ its t -conorm, and $\Rightarrow$ is the residuum of T :

$$a \Rightarrow b := \sup\{c \in L \mid T(a, c) \leq b\}.$$

Write $T_s(\alpha_1, \dots, \alpha_s)$ for the iterated t -norm ($T_1(\alpha) = \alpha$, $T_s(\vec{\alpha}) = T(\alpha_1, T_{s-1}(\alpha_2, \dots, \alpha_s))$), and $\bigvee$ for the supremum in L .

Definition 9.3 (Fuzzy Philosophy). (cf. [147–150]) Fix $\mathbf{L}$ as in Theorem 9.2. Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ an address map. Let M be a nonempty set of *methods* and R a nonempty set of *contexts*. For each $(m, r) \in M \times R$ let

$$\mathcal{R}_{m,r} \subseteq \bigcup_{s \geq 1} (A^s \times A) \times L$$

be a (possibly infinite) set of *fuzzy rules*; we write $\rho : (a_1, \dots, a_s) \Rightarrow b$ with strength $\lambda_\rho \in L$.

A *fuzzy state* is a map $\mu \in L^A$. The *one-step fuzzy derivation* at (m, r) is

$$\text{Der}_{m,r}^F(\mu)(b) := S\left(\mu(b), \bigvee_{\rho \in \mathcal{R}_{m,r}} \bigvee_{\rho : (a_1, \dots, a_s) \Rightarrow b} T(\lambda_\rho, T_s(\mu(a_1), \dots, \mu(a_s)))\right).$$

From an initial state μ_0 (e.g. the indicator of premises $\Gamma \subseteq A$), define

$$\mu_{t+1} := \text{Der}_{m,r}^F(\mu_t) \quad (t \geq 0), \quad \text{Cn}_{m,r}^F(\mu_0) := \bigvee_{t \geq 0} \mu_t \in L^A.$$

For $q \in Q$, the (degree-valued) set of *acceptable answers* is the restriction

$$\text{Ans}^F(q \mid \mu_0; m, r)(a) := \begin{cases} \text{Cn}_{m,r}^F(\mu_0)(a), & \text{adr}(a) = q, \\ 0, & \text{otherwise.} \end{cases}$$

The tuple

$$\mathbf{FP} := (A, Q, \text{adr}, M, R, \mathbf{L}, \{\mathcal{R}_{m,r}\}_{(m,r)})$$

is called a *Fuzzy Philosophy*. For every fixed (m, r) , $\text{Cn}_{m,r}^F$ is an extensive, monotone, idempotent operator on L^A (a Tarskian closure), and equals the least fixed point of $\text{Der}_{m,r}^F$ above μ_0 .

Example 9.4 (Fuzzy Philosophy in practice: deciding on a remote-work policy). Fix the truth-degree lattice $L = [0, 1]$ with $T = \min$ (t-norm), $S = \max$ (s-norm), and $T_s = \min$ for s -ary conjunctions. Consider the question

$$Q = \{q_{\text{policy}}\} \quad (\text{“Is adopting a remote-work policy ethically justified?”})$$

and the set of propositions

$$A = \{R, P, W, C, E, \bar{E}\},$$

with meanings:

- R : “The policy reduces commuting emissions (environmental benefit).”
- P : “The policy maintains (or improves) productivity.”
- W : “The policy supports employees’ work-life balance.”
- C : “The policy harms collaboration and mentoring.”
- E : “Adopting the policy is ethically justified.” (answer to q_{policy})
- $\bar{E}$: “Adopting the policy is *not* ethically justified.” (answer to q_{policy})

Let the address map send only answers to the question:

$$\text{adr}(E) = \text{adr}(\bar{E}) = q_{\text{policy}}, \quad \text{adr}(R) = \text{adr}(P) = \text{adr}(W) = \text{adr}(C) = q_{\text{policy}} \text{ (auxiliary facts).}$$

Fix one method $m^\star$ (“utilitarian balancing”) and one context $r^\star$ (“urban tech firm”). Provide the following fuzzy rules (with strengths $\lambda_\rho \in [0, 1]$):

$$\begin{aligned} \rho_1 : (R, P) &\Rightarrow E && \text{with } \lambda_{\rho_1} = 0.8, \\ \rho_2 : (W) &\Rightarrow E && \text{with } \lambda_{\rho_2} = 0.6, \\ \rho_3 : (C) &\Rightarrow \bar{E} && \text{with } \lambda_{\rho_3} = 0.7. \end{aligned}$$

Write $\mathcal{R}_{m^*, r^*} = \{\rho_1, \rho_2, \rho_3\}$, and let the one-step fuzzy derivation be

$$\text{Der}_{m^*, r^*}^F(\mu)(b) = S\left(\mu(b), \bigvee_{\rho: (a_1, \dots, a_s) \Rightarrow b} T(\lambda_\rho, T_s(\mu(a_1), \dots, \mu(a_s)))\right)$$

with $S = \max$, $T = \min$, $T_s = \min$.

Initial evidence. Assume the firm's data (scaled to $[0, 1]$) give the fuzzy state

$$\mu_0(R) = 0.9, \quad \mu_0(P) = 0.7, \quad \mu_0(W) = 0.6, \quad \mu_0(C) = 0.4, \quad \mu_0(E) = 0, \quad \mu_0(\bar{E}) = 0.$$

One-step derivation (numerical). For E :

$$\begin{aligned} \text{via } \rho_1 : \quad & T_s(\mu_0(R), \mu_0(P)) = \min(0.9, 0.7) = 0.7, \\ & T(\lambda_{\rho_1}, \cdot) = \min(0.8, 0.7) = 0.7; \\ \text{via } \rho_2 : \quad & T_s(\mu_0(W)) = 0.6, \quad T(\lambda_{\rho_2}, \cdot) = \min(0.6, 0.6) = 0.6; \\ & \bigvee (\dots) = \max(0.7, 0.6) = 0.7, \quad \text{Der}_{m^*, r^*}^F(\mu_0)(E) = S(\mu_0(E), 0.7) = \max(0, 0.7) = 0.7. \end{aligned}$$

For $\bar{E}$:

$$\text{via } \rho_3 : \quad T_s(\mu_0(C)) = 0.4, \quad T(\lambda_{\rho_3}, \cdot) = \min(0.7, 0.4) = 0.4,$$

so

$$\text{Der}_{m^*, r^*}^F(\mu_0)(\bar{E}) = S(\mu_0(\bar{E}), 0.4) = \max(0, 0.4) = 0.4.$$

Thus the next state is

$$\mu_1(E) = 0.7, \quad \mu_1(\bar{E}) = 0.4,$$

and all other coordinates unchanged.

Fixed point and answers. A second iteration leaves μ unchanged (the premises are fixed and $S = \max$), hence $\text{Cn}_{m^*, r^*}^F(\mu_0) = \mu_1$. Therefore, the degree-valued set of acceptable answers to q_{policy} is

$$\text{Ans}^F(q_{\text{policy}} \mid \mu_0; m^*, r^*)(E) = 0.7, \quad \text{Ans}^F(q_{\text{policy}} \mid \mu_0; m^*, r^*)(\bar{E}) = 0.4.$$

If a crisp decision is required, one may, for example, pick the maximum-degree answer (here E), or apply a separate defuzzification threshold.

10 Neutrosophic Set and Neutrosophy

Neutrosophy extends classical and fuzzy logics by allowing truth, indeterminacy, and falsity to vary independently within a nonstandard interval [4, 152]. A *Neutrosophic Set* assigns to each element independent degrees of truth, indeterminacy, and falsity, all ranging within the extended unit interval [153–156]. Related notions include the *Bipolar Neutrosophic Set* [157–159], *Double-valued Neutrosophic Set* [160–163], *Quadri-Partitioned Neutrosophic Set* [164, 165], *Pentapartitioned Neutrosophic Set* [166–168], and *HeptaPartitioned Neutrosophic Set* [169], each representing structured generalizations of neutrosophic reasoning. Moreover, the ideas of Neutrosophy have been applied to a wide range of concepts across diverse fields, such as Graph Theory [170–173], Algebra [174–176], and Decision-Making [177, 178].

Definition 10.1 (Neutrosophic Set). [4, 152] Let X be a non-empty set. A *Neutrosophic Set* (NS) A on X is characterized by three membership functions:

$$T_A : X \rightarrow [0, 1], \quad I_A : X \rightarrow [0, 1], \quad F_A : X \rightarrow [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 10.2 (Extended unit interval). Fix a nonstandard extension $\mathbb{R}^*$ of the real numbers. The *extended unit interval* is

$$]^{-0}, 1^{+}[:= \{r \in \mathbb{R}^* \mid -0 \leq r \leq 1^{+}\},$$

which permits infinitesimal undershoot and overshoot at the endpoints 0 and 1.

Definition 10.3 (Neutrosophic valuation). Let **Prop** be a nonempty set of propositions and **Ref** a class of *contexts*. A *neutrosophic valuation* is a function

$$v : \text{Prop} \times \text{Ref} \longrightarrow (\mathcal{P}(]^{-0}, 1^{+}[))^3, \quad (A, R) \longmapsto (T_R(A), I_R(A), F_R(A)),$$

where $T_R(A), I_R(A), F_R(A) \subseteq]^{-0}, 1^{+}[$ represent, respectively, the degrees (possibly as ranges) of truth, indeterminacy, and falsity of proposition A in context R .

These three components are *independent*: no constraint such as $t + i + f = 1$ or $t + i + f \leq 1$ is required. One may have $t + i + f < 1$, $t + i + f = 1$, or $t + i + f > 1$ (up to 3^{+}), for admissible $t \in T_R(A)$, $i \in I_R(A)$, $f \in F_R(A)$.

Definition 10.4 (Neutrosophy). [4, 152] *Neutrosophy* is the study of such valuations together with the *neutrality schema*: for every idea A , one considers its opposite $\text{Anti-}A$, its complement $\text{Non-}A$, and the continuum of neutralities $\text{Neut-}A$ lying between A and $\text{Anti-}A$.

(Fundamental Thesis) Every idea A admits context-dependent degrees of being true, indeterminate, and false, captured by the triple

$$(T_R(A), I_R(A), F_R(A)) \subseteq]^{-0}, 1^{+}[^3.$$

Example 10.5 (Neutrosophy in daily life: “The package will arrive today”). Let the idea be

$$A : \text{“The package will arrive today.”}$$

According to the neutrality schema, we also consider its opposite $\text{Anti-}A$: “The package will *not* arrive today”, its complement $\text{Non-}A$ (any outcome not equal to A), and the continuum of neutralities $\text{Neut-}A$ between A and $\text{Anti-}A$.

Contextual evidence (sources). Take three contexts/sources:

$$R = \{\text{carrier tracking, weather alert, hub congestion}\}.$$

Assign single-valued neutrosophic assessments $(T, I, F) \in [0, 1]^3$ to A from each source:

$$\begin{aligned} \text{Carrier tracking (out for delivery)} & : (T, I, F) = (0.85, 0.10, 0.05), \\ \text{Severe weather alert} & : (0.30, 0.20, 0.50), \\ \text{Sorting hub congestion news} & : (0.25, 0.30, 0.45). \end{aligned}$$

Weight the sources by trust $(w_1, w_2, w_3) = (0.6, 0.2, 0.2)$ (summing to 1). The aggregated neutrosophic valuation of A is the weighted average:

$$\begin{aligned} T(A) &= 0.6 \cdot 0.85 + 0.2 \cdot 0.30 + 0.2 \cdot 0.25 = 0.51 + 0.06 + 0.05 = 0.62, \\ I(A) &= 0.6 \cdot 0.10 + 0.2 \cdot 0.20 + 0.2 \cdot 0.30 = 0.06 + 0.04 + 0.06 = 0.16, \\ F(A) &= 0.6 \cdot 0.05 + 0.2 \cdot 0.50 + 0.2 \cdot 0.45 = 0.03 + 0.10 + 0.09 = 0.22. \end{aligned}$$

Thus $v(A) = (T, I, F) = (0.62, 0.16, 0.22)$ (note $T + I + F = 1$ here, but no global normalization is required).

Opposite and complement (illustrative mappings). A common neutrosophic reflection of the opposite idea is to swap truth and falsity while preserving indeterminacy:

$$v(\text{Anti-}A) := (F(A), I(A), T(A)) = (0.22, 0.16, 0.62).$$

One convenient way to *illustrate* a complement valuation is to split the indeterminacy evenly between A and its opposite; e.g.

$$v(\text{Non-}A) := (F(A) + \tfrac{1}{2}I(A), I(A), T(A) - \tfrac{1}{2}I(A)) = (0.22 + 0.08, 0.16, 0.62 - 0.08) = (0.30, 0.16, 0.54).$$

(Other context-dependent complement conventions are possible in neutrosophy.)

Decision hint. Since $T(A) - F(A) = 0.62 - 0.22 = 0.40 > 0$, the current evidence favors A (arrival today), while $I(A) = 0.16$ explicitly records remaining uncertainty (e.g. weather or operational disruptions).

11 Plithogenic Set and Plithogeny (Plithogenic-Philosophy)

A Plithogenic Set generalizes fuzzy, intuitionistic, and neutrosophic sets by incorporating attributes, values, and contradiction degrees into aggregation [22, 179]. Plithogenic-Philosophy extends philosophy by aggregating propositions with attributes, using contradiction degrees to blend fuzzy, intuitionistic, and neutrosophic reasoning (cf. [22, 179]). Applications of Plithogenic concepts are known in Plithogenic Graphs [180, 181], Plithogenic Algebra [182–184], and Plithogenic Decision-Making [185].

Definition 11.1 (Plithogeny (conceptual)). Plithogeny is the genesis and evolution of new entities by the dynamic, organic fusion of many kinds of opposites, their neutrals, and non-opposites. It strictly generalizes both dialectics (one pair of opposites) and neutrosophy (one pair of opposites together with its neutrality), by allowing multiple, possibly interacting pairs with neutralities and other non-opposed factors.

Definition 11.2 (Plithogenic-Philosophy (formal, single attribute)). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ an address map. Fix a (possibly multi)attribute α and a nonempty set V of its values (“perspectives”, criteria, contexts).

Degrees. Choose $z \in \{1, 2, 3\}$ and a degree codomain

$$D = \begin{cases} [0, 1] & (z = 1, \text{fuzzy}), \\ [0, 1]^2 & (z = 2, \text{intuitionistic: membership/nonmembership}), \\ [0, 1]^3 & (z = 3, \text{neutrosophic: truth/indeterminacy/falsehood}). \end{cases}$$

A *plithogenic valuation* is a map

$$d : A \times V \longrightarrow D, \quad (a, v) \longmapsto d(a \mid v),$$

assigning to each proposition a a degree (or tuple of degrees) *relative to* the attribute value v .

Contradiction & dominance. Let $c : V \times V \rightarrow [0, 1]$ be a symmetric *contradiction (dissimilarity) degree* with $c(v, v) = 0$. Optionally fix a context-dependent *dominant value* $v_D \in V$. Write $c_v := c(v_D, v)$.

T-norm/T-conorm. Fix a left-continuous t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ and its residuated t -conorm $S : [0, 1]^2 \rightarrow [0, 1]$.

Plithogenic aggregation on degrees. For $x, y \in [0, 1]$ and $c \in [0, 1]$, define the basic plithogenic mixer

$$\text{Agg}_c(x, y) := (1 - c)T(x, y) + cS(x, y).$$

- *Fuzzy case* ($z = 1$). Let $\mu_a(v) := d(a \mid v) \in [0, 1]$. For $a, b \in A$ and $v \in V$,

$$\mu_{a \wedge_P b}(v) := \text{Agg}_{c_v}(\mu_a(v), \mu_b(v)), \quad \mu_{a \vee_P b}(v) := \text{Agg}_{1-c_v}(\mu_a(v), \mu_b(v)),$$

so that c_v linearly interpolates between T and S (and swaps roles for $\vee_P$).

- *Intuitionistic case* ($z = 2$). Write $d(a \mid v) = (\mu_a(v), \nu_a(v))$ with $\mu, \nu \in [0, 1]$, typically $\mu + \nu \leq 1$. Define

$$(\mu, \nu)_{a \wedge_P b}(v) := \left(\text{Agg}_{c_v}(\mu_a, \mu_b), \text{Agg}_{1-c_v}(\nu_a, \nu_b) \right),$$

$$(\mu, \nu)_{a \vee_P b}(v) := \left(\text{Agg}_{1-c_v}(\mu_a, \mu_b), \text{Agg}_{c_v}(\nu_a, \nu_b) \right).$$

- *Neutrosophic case* ($z = 3$). Write $d(a \mid v) = (t_a(v), i_a(v), f_a(v)) \in [0, 1]^3$ without sum constraint. Set

$$(t, i, f)_{a \wedge_P b}(v) := \left(\text{Agg}_{c_v}(t_a, t_b), \frac{1}{2} [\text{Agg}_{c_v}(i_a, i_b) + \text{Agg}_{1-c_v}(i_a, i_b)], \text{Agg}_{1-c_v}(f_a, f_b) \right),$$

$$(t, i, f)_{a \vee_P b}(v) := \left(\text{Agg}_{1-c_v}(t_a, t_b), \frac{1}{2} [\text{Agg}_{c_v}(i_a, i_b) + \text{Agg}_{1-c_v}(i_a, i_b)], \text{Agg}_{c_v}(f_a, f_b) \right).$$

The indeterminacy component takes the mean of the two mixes (a standard choice consistent with viewing i as halfway between t and f).

Answers under attributes. Given premises $\Gamma \subseteq A$, one may compute attribute-wise aggregates (e.g. finite $\wedge_P/\vee_P$ -folds) to obtain degrees for each $a \in A$ and $v \in V$, then report the acceptable answers to $q \in Q$ by restricting to $\text{adr}^{-1}(\{q\})$.

Example 11.3 (Plithogenic-Philosophy in practice: selecting a workplace policy). *Question.* $Q = \{\text{Workplace policy for this year}\}$
Propositions.

$$A = \{a_1 : \text{“Remote 3 days/week”}, a_2 : \text{“Office every day”}, a_3 : \text{“Hybrid (2 office days)”}\},$$

with $\text{adr}(a_i) = Q$ for all i .

Attribute (perspectives). Let the single attribute be $\alpha = \text{“perspective”}$ with value set

$$V = \{v_C = \text{collaboration}, v_P = \text{productivity}, v_K = \text{cost}\}.$$

Fix the dominant value $v_D = v_C$. Take the contradiction (dissimilarity) degrees (symmetric)

$$c(v_D, v_C) = 0, \quad c(v_D, v_P) = 0.3, \quad c(v_D, v_K) = 0.8.$$

Degree model and mixer. Use the fuzzy case ($z = 1$) with degrees $\mu \in [0, 1]$. Take $T(x, y) = \min\{x, y\}$ and $S(x, y) = \max\{x, y\}$. For $c \in [0, 1]$ define the plithogenic aggregator

$$\text{Agg}_c(x, y) := (1 - c)T(x, y) + cS(x, y).$$

To aggregate a proposition a across perspectives, fold from the dominant value v_D :

$$\mu_a^{(v_D)} := \mu_a(v_D), \quad \mu_a^{(v_D \rightarrow v)} := \text{Agg}_{c(v_D, v)}(\mu_a^{(v_D)}, \mu_a(v)),$$

and apply the fold sequentially for $v \in \{v_P, v_K\}$.

Expert assessments (by perspective).

Proposition	$\mu(\cdot \mid v_C)$ (collab)	$\mu(\cdot \mid v_P)$ (prod)	$\mu(\cdot \mid v_K)$ (cost)
a_1 : Remote 3d	0.60	0.80	0.90
a_2 : Office daily	0.90	0.60	0.30
a_3 : Hybrid 2d	0.85	0.75	0.60

Plithogenic aggregation (dominant v_C).

For a_1 (Remote 3d): first fold v_P with $c = 0.3$,

$$\text{Agg}_{0.3}(0.60, 0.80) = (1 - 0.3) \min(0.60, 0.80) + 0.3 \max(0.60, 0.80) = 0.7 \cdot 0.60 + 0.3 \cdot 0.80 = 0.42 + 0.24 = 0.66.$$

Then fold v_K with $c = 0.8$,

$$\text{Agg}_{0.8}(0.66, 0.90) = 0.2 \cdot \min(0.66, 0.90) + 0.8 \cdot \max(0.66, 0.90) = 0.2 \cdot 0.66 + 0.8 \cdot 0.90 = 0.132 + 0.72 = 0.852.$$

For a_2 (Office daily): fold v_P with $c = 0.3$,

$$\text{Agg}_{0.3}(0.90, 0.60) = 0.7 \cdot 0.60 + 0.3 \cdot 0.90 = 0.42 + 0.27 = 0.69,$$

then v_K with $c = 0.8$,

$$\text{Agg}_{0.8}(0.69, 0.30) = 0.2 \cdot 0.30 + 0.8 \cdot 0.69 = 0.06 + 0.552 = 0.612.$$

For a_3 (Hybrid 2d): fold v_P with $c = 0.3$,

$$\text{Agg}_{0.3}(0.85, 0.75) = 0.7 \cdot 0.75 + 0.3 \cdot 0.85 = 0.525 + 0.255 = 0.780,$$

then v_K with $c = 0.8$,

$$\text{Agg}_{0.8}(0.780, 0.60) = 0.2 \cdot 0.60 + 0.8 \cdot 0.780 = 0.120 + 0.624 = 0.744.$$

Outcome (ranked answers to the question).

$$\mu(a_1) = 0.852 \quad > \quad \mu(a_3) = 0.744 \quad > \quad \mu(a_2) = 0.612.$$

Under the chosen dominant perspective (collaboration) and contradictions ($c(v_C, v_P) = 0.3$, $c(v_C, v_K) = 0.8$), the plithogenic aggregation yields the preference order:

$$\text{Remote 3 days/week} \succ \text{Hybrid 2 days} \succ \text{Office daily}.$$

Intuitively, the high contradiction between collaboration and cost ($c = 0.8$) tilts the final mix toward the *t-conorm* effect on the cost dimension, which favors policies with strong cost performance (here, remote work).

12 Functorial Structure and Functorial-Philosophy

Functorial Structure is a covariant functor assigning each object a structured set, with morphisms inducing structure-preserving pushforward operations [14]. Functorial-Philosophy is a functor from contextual categories to philosophical structures, transporting questions, propositions, and consequence relations consistently across morphisms.

Definition 12.1 (Functorial Set). [14] Let C be a category and

$$F: C \longrightarrow \mathbf{Set}$$

be a (covariant) endofunctor. For any object $X \in \text{Ob}(C)$, an *F-set over X* is an element

$$s \in F(X).$$

We denote the collection of all *F*-sets over X simply by $F(X)$. A morphism $f: X \rightarrow Y$ in C induces a *pushforward*

$$F(f): F(X) \longrightarrow F(Y), \quad s \mapsto F(f)(s).$$

Definition 12.2 (Functorial Structure). [14] Let $\mathbb{C}$ be a category. A *Functorial Structure* on $\mathbb{C}$ is simply a covariant functor

$$F: \mathbb{C} \longrightarrow \mathbf{Set}.$$

For each object $X \in \text{Ob}(\mathbb{C})$, an element

$$s \in F(X)$$

is called an *F-structure on X*. Every morphism $f: X \rightarrow Y$ in $\mathbb{C}$ induces a *pushforward*

$$F(f): F(X) \longrightarrow F(Y), \quad s \mapsto F(f)(s),$$

and the usual functoriality conditions $F(\text{id}_X) = \text{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

Example 12.3 (Document format conversions (Functorial Set/Structure)). Let $\mathbb{C}$ be the category whose objects are file formats (Markdown, HTML, PDF, ...) and whose morphisms are format converters $f: X \rightarrow Y$. Define a functor $F: \mathbb{C} \rightarrow \mathbf{Set}$ by $F(X) = \{\text{documents stored in format } X\}$ and $F(f) = \text{“apply the converter to the document”}$. If $s = \text{minutes.md} \in F(\text{MD})$ and $f: \text{MD} \rightarrow \text{PDF}$, then

$$F(f)(s) = \text{minutes.pdf}, \quad F(g \circ f) = F(g) \circ F(f) \text{ (compose converters)}.$$

Example 12.4 (Time-zone scheduling (Functorial Set/Structure)). Let $\mathbb{C}$ be the category of time zones; a morphism $f: X \rightarrow Y$ converts local timestamps $X \rightarrow Y$. Define $F(X) = \{\text{meeting slots scheduled in zone } X\}$ and $F(f)$ converts each slot to zone Y . For $s = \text{“Aug 30, 10:00 in JST”} \in F(\text{JST})$ and $f: \text{JST} \rightarrow \text{PST}$,

$$F(f)(s) = \text{“Aug 29, 18:00 in PST”},$$

$$F(\text{id}) = \text{id}, \quad F(g \circ f) = F(g) \circ F(f).$$

Example 12.5 (Product taxonomy remapping (Functorial Set/Structure)). Objects of $\mathbb{C}$ are category taxonomies; a morphism $f: X \rightarrow Y$ maps each category in X to one in Y . Let $F(X) = \{\text{SKU-category assignments under taxonomy } X\}$. Then $F(f)$ reclassifies each SKU by sending its category along f :

$$(sku, \text{cat}_X) \mapsto (sku, f(\text{cat}_X)).$$

Functoriality captures that successive remaps compose.

Example 12.6 (Map projections for GPS traces (Functorial Set/Structure)). Let $\mathbb{C}$ be the category of coordinate systems; a morphism $g: X \rightarrow Y$ is a coordinate transform. Define $F(X) = \{\text{GPS traces in } X\}$ where a trace is a finite sequence of points in X . Set $F(g)$ to act pointwise on traces:

$$F(g)((p_1, \dots, p_n)) = (g(p_1), \dots, g(p_n)).$$

Then $F(\text{id}) = \text{id}$ and $F(g \circ f) = F(g) \circ F(f)$ hold by construction.

Definition 12.7 (Category of philosophical structures). A *philosophical structure* is a tuple

$$\mathbf{P} = (Q, A, \text{adr}, \text{Cn}),$$

where Q is a nonempty set of *questions*, A a nonempty set of *propositions*, $\text{adr}: A \rightarrow Q$ an *address map*, and $\text{Cn}: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ a Tarskian consequence operator (extensive, monotone, idempotent).

A *morphism of philosophical structures* $h: \mathbf{P} \rightarrow \mathbf{P}'$ is a pair of maps $h_Q: Q \rightarrow Q'$ and $h_A: A \rightarrow A'$ such that

$$\text{adr}' \circ h_A = h_Q \circ \text{adr} \quad \text{and} \quad h_A[\text{Cn}(\Gamma)] \subseteq \text{Cn}'(h_A[\Gamma]) \quad \text{for all } \Gamma \subseteq A.$$

With objectwise identities and composition $(k \circ h)_A := k_A \circ h_A$, $(k \circ h)_Q := k_Q \circ h_Q$, these form a category

PhilStr.

Definition 12.8 (Functorial-Philosophy). Let C be a category of *contexts* (e.g. times, agents, domains, datasets). A *Functorial-Philosophy over C* is a covariant functor

$$\mathbf{P}: C \longrightarrow \mathbf{PhilStr},$$

that assigns to each object $X \in C$ a philosophical structure

$$\mathbf{P}(X) = (Q_X, A_X, \text{adr}_X, \text{Cn}_X),$$

and to each morphism $f: X \rightarrow Y$ a structure map

$$\mathbf{P}(f) = (f_Q: Q_X \rightarrow Q_Y, f_A: A_X \rightarrow A_Y)$$

satisfying, for all $\Gamma \subseteq A_X$,

$$\text{adr}_Y \circ f_A = f_Q \circ \text{adr}_X, \quad f_A[\text{Cn}_X(\Gamma)] \subseteq \text{Cn}_Y(f_A[\Gamma]),$$

together with functoriality $\mathbf{P}(\text{id}_X) = \text{id}_{\mathbf{P}(X)}$ and $\mathbf{P}(g \circ f) = \mathbf{P}(g) \circ \mathbf{P}(f)$.

Example 12.9 (Functorial-Philosophy in practice: escalating medical triage from telemedicine to the emergency department). Let C be the small category with two objects

$$T = \text{“Telemedicine encounter”}, \quad E = \text{“Emergency Department (ED) encounter”}$$

and a single non-identity morphism $u: T \rightarrow E$ (“escalation”), besides identities.

Define a functorial philosophy

$$\mathbf{P}: C \longrightarrow \mathbf{PhilStr}, \quad X \mapsto \mathbf{P}(X) = (Q_X, A_X, \text{adr}_X, \text{Cn}_X),$$

that maps each clinical *context* $X \in \{T, E\}$ to a philosophical structure of questions, propositions, addressing, and consequence.

Telemedicine context T .

- Questions:

$$Q_T = \{ q_{\text{disp}} = \text{"What disposition?"}, q_{\text{obs}} = \text{"What key finding?"} \}.$$

- Propositions (answers / findings):

$$A_T = \{ s = \text{"Self-care at home"}, c = \text{"Clinic within 24h"}, e = \text{"Call EMS now"}, \\ rf = \text{"Red flag present"}, nf = \text{"No red flags"} \}.$$

- Address map:

$$\text{adr}_T(s) = \text{adr}_T(c) = \text{adr}_T(e) = q_{\text{disp}}, \\ \text{adr}_T(rf) = \text{adr}_T(nf) = q_{\text{obs}}.$$

- Consequence operator $\text{Cn}_T : \mathcal{P}(A_T) \rightarrow \mathcal{P}(A_T)$ (Tarskian closure) generated by the rules

$$rf \Rightarrow e, \quad nf \Rightarrow c,$$

i.e. for any $\Gamma \subseteq A_T$,

$$\text{Cn}_T(\Gamma) := \Gamma \cup \{ e \mid rf \in \Gamma \} \cup \{ c \mid nf \in \Gamma \}.$$

ED context E .

- Questions:

$$Q_E = \{ q'_{\text{tri}} = \text{"ED triage category?"}, q'_{\text{obs}} = \text{"What key finding?"} \}.$$

- Propositions:

$$A_E = \{ t_1 = \text{"Immediate (ESI-1)"}, t_3 = \text{"Urgent (ESI-3)"}, \\ rf' = \text{"Red flag present (ED)"}, nf' = \text{"No red flags (ED)} \}.$$

- Address map:

$$\text{adr}_E(t_1) = \text{adr}_E(t_3) = q'_{\text{tri}}, \\ \text{adr}_E(rf') = \text{adr}_E(nf') = q'_{\text{obs}}.$$

- Consequence operator Cn_E generated by

$$rf' \Rightarrow t_1, \quad nf' \Rightarrow t_3,$$

i.e. for any $\Delta \subseteq A_E$,

$$\text{Cn}_E(\Delta) := \Delta \cup \{ t_1 \mid rf' \in \Delta \} \cup \{ t_3 \mid nf' \in \Delta \}.$$

Action on the escalation morphism $u : T \rightarrow E$. Define $P(u) = (u_Q, u_A)$ by

$$u_Q(q_{\text{disp}}) = q'_{\text{tri}}, \quad u_Q(q_{\text{obs}}) = q'_{\text{obs}}, \\ u_A(s) = t_3, \quad u_A(c) = t_3, \quad u_A(e) = t_1, \quad u_A(rf) = rf', \quad u_A(nf) = nf'.$$

Then the functoriality side-conditions hold:

$$\text{adr}_E \circ u_A = u_Q \circ \text{adr}_T \quad (\text{by inspection}),$$

and for all $\Gamma \subseteq A_T$,

$$u_A[\text{Cn}_T(\Gamma)] \subseteq \text{Cn}_E(u_A[\Gamma]) \quad (\text{closure monotonicity}).$$

Indeed, if $rf \in \Gamma$ then $e \in \text{Cn}_T(\Gamma)$ and $u_A(e) = t_1 \in \text{Cn}_E(\{u_A(rf) = rf'\})$; similarly, if $nf \in \Gamma$ then $c \in \text{Cn}_T(\Gamma)$ and $u_A(c) = t_3 \in \text{Cn}_E(\{u_A(nf) = nf'\})$. Trivial inclusions hold for premises, hence the containment follows.

Two concrete patient scenarios.

(a) *Chest pain with a red flag on telemedicine.* Premises $\Gamma_1 = \{rf\} \subseteq A_T$. Then

$$\text{Cn}_T(\Gamma_1) = \{rf, e\} \Rightarrow u_A[\text{Cn}_T(\Gamma_1)] = \{rf', t_1\} \subseteq \text{Cn}_E(\{rf'\}) = \{rf', t_1\}.$$

Interpretation: “red flag” escalates to ED and functorially becomes “immediate triage”.

(b) *Minor complaint without red flags on telemedicine.* Premises $\Gamma_2 = \{nf\} \subseteq A_T$. Then

$$\text{Cn}_T(\Gamma_2) = \{nf, c\} \Rightarrow u_A[\text{Cn}_T(\Gamma_2)] = \{nf', t_3\} \subseteq \text{Cn}_E(\{nf'\}) = \{nf', t_3\}.$$

Interpretation: “no red flags” maps to a non-immediate ED category if escalation occurs (e.g. after-hours routing).

The assignment $X \mapsto \mathbf{P}(X)$ with $\mathbf{P}(u)$ as above defines a covariant functor $\mathbf{P} : \mathcal{C} \rightarrow \mathbf{PhilStr}$: it transports questions and propositions across care settings while preserving addresses and respecting consequence (reasoning) via $u_A[\text{Cn}_T(\cdot)] \subseteq \text{Cn}_E(u_A[\cdot])$. This captures, in a real workflow, how decisions and their justifications functorially propagate from telemedicine screening to in-hospital triage.

13 Anti-Structure and Anti-Philosophy

Anti-Philosophy rejects ill-posed problems through the use of partial semantics, focusing exclusively on operationally meaningful states while dissolving invalid formulations (cf. [186–188]). Related concepts such as *Anti-Algebra* are also known [189, 190].

Definition 13.1 (Anti-Structure (partial, feasibility-based algebra over a signature)). Fix a single-sorted, finitary signature

$$\Sigma = (\text{Func}, \text{Rel}, \text{ar}_{\text{Func}}, \text{ar}_{\text{Rel}}).$$

An *Anti-Structure* over Σ is a tuple

$$\mathbf{A}^{\text{anti}} = \left(H, (f^\#, \text{Feas}_f)_{f \in \text{Func}}, (R^\#, \text{Feas}_R)_{R \in \text{Rel}} \right),$$

where:

1. $H \neq \emptyset$ is the carrier set.
2. For each function symbol f of arity m , $\text{Feas}_f \subseteq H^m$ is a *feasibility domain* and $f^\# : \text{Feas}_f \rightarrow H$ is a (partial) operation; inputs in $H^m \setminus \text{Feas}_f$ are *dissolved* (left undefined).
3. For each relation symbol R of arity r , $\text{Feas}_R \subseteq H^r$ is a feasibility domain and $R^\# \subseteq \text{Feas}_R$ is the *admissible* part of R ; tuples in $H^r \setminus \text{Feas}_R$ are dissolved.

Evaluation of composite terms is defined iff every intermediate tuple lies in the corresponding feasibility set; otherwise the value is undefined.

Example 13.2 (Anti-Structure in practice: feasibility-driven card payments). Consider the single-sorted signature Σ with one ternary function symbol `pay` and one ternary relation symbol `auth`. Let the carrier H be the disjoint union

$$H = \{c1, c2\} \uplus \{mA, mB\} \uplus \{\$120, \$700\} \uplus \{rOK, rDecl\},$$

encoding two cards, two merchants, two amounts, and two receipts.

Feasibility domains. Define the feasibility set for the function **pay** by the policy

$$\text{Feas}_{\text{pay}} = \left\{ (c, m, a) \in H^3 \mid \begin{array}{l} c \in \{c1, c2\}, m \in \{mA, mB\}, a \in \{\$120, \$700\}, \\ \text{card } c \text{ is active, } m \text{ not blocked, and } a \text{ is within card limit} \end{array} \right\}.$$

Concretely, assume: card $c1$ has limit \$500 and is active; card $c2$ has limit \$1000 and is active; merchant mA is unblocked, mB is blocked. Then

$$(c1, mA, \$120), (c2, mA, \$700) \in \text{Feas}_{\text{pay}}, \quad (c1, mA, \$700), (c1, mB, \$120) \notin \text{Feas}_{\text{pay}}.$$

Define the partial operation $\text{pay}^\# : \text{Feas}_{\text{pay}} \rightarrow H$ by

$$\text{pay}^\#(c1, mA, \$120) = \text{rOK}, \quad \text{pay}^\#(c2, mA, \$700) = \text{rOK},$$

and leave $\text{pay}^\#$ undefined outside Feas_{pay} (requests are *dissolved*).

For the relation **auth**, let

$$\text{Feas}_{\text{auth}} = \{ (c, m, a) \in H^3 : c \in \{c1, c2\}, m \in \{mA, mB\}, a \in \{\$120, \$700\} \},$$

and $\text{auth}^\# \subseteq \text{Feas}_{\text{auth}}$ be the admissible (approved) triples, here

$$\text{auth}^\# = \{ (c1, mA, \$120), (c2, mA, \$700) \}.$$

Interpretation. The anti-structure $\mathbf{A}^{\text{anti}}$ thus *refuses to interpret* ill-posed or infeasible calls such as $\text{pay}(c1, mA, \$700)$ (over limit) or $\text{pay}(c1, mB, \$120)$ (blocked merchant): these inputs fall outside Feas_{pay} and are dissolved. Only operationally meaningful inputs (feasible authorizations) are evaluated and produce receipts in H .

Definition 13.3 (Operational semantics and dissolution). Let S be a nonempty set (*states/observations*). Given a philosophical structure $\mathbf{P} = (\delta_{\mathbf{P}}, \text{Cn}_{\mathbf{P}})$, an *operational semantics* is a *partial map*

$$\text{Semp}_{\mathbf{P}} : W \rightarrow \mathcal{P}(S),$$

defined on a domain $\text{Dom}(\text{Semp}_{\mathbf{P}}) \subseteq W$. Elements outside $\text{Dom}(\text{Semp}_{\mathbf{P}})$ are deemed *ill-posed* and are *dissolved* (ignored).

Definition 13.4 (Anti-philosophical consequence). For $\Gamma \subseteq W$ define the *anti-philosophical closure* on the operational domain by

$$\begin{aligned} \text{Cn}_{\mathbf{P}}^{\text{A}}(\Gamma) &:= \left\{ a \in \text{Dom}(\text{Semp}_{\mathbf{P}}) \right. \\ &\quad \left. \mid \bigcap_{g \in \Gamma \cap \text{Dom}(\text{Semp}_{\mathbf{P}})} \text{Semp}_{\mathbf{P}}(g) \subseteq \text{Semp}_{\mathbf{P}}(a) \right\}. \end{aligned}$$

Answers to a query q (if a question map $\text{adr} : W \rightarrow Q$ is present) are

$$\begin{aligned} &\text{Ans}_{\mathbf{P}}^{\text{A}}(q \mid \Gamma) \\ &:= \text{Cn}_{\mathbf{P}}^{\text{A}}(\Gamma) \cap \text{adr}^{-1}(\{q\}) \end{aligned}$$

.

Example 13.5 (Anti-Philosophy in practice: helpdesk triage with partial semantics). Let

$$S = \{\text{check_router}, \text{escalate_NOC}\}$$

be the set of concrete actions a network helpdesk can take. Consider the set of utterances (the “world” of questions/propositions)

$$W = \{ w_1 = \text{“No internet across multiple rooms”}, w_2 = \text{“Building-wide outage suspected”},$$

$$a^* = \text{“Open network incident ticket”}, w_{\text{ill}} = \text{“What is the meaning of life?”} \}.$$

Define a *partial* operational semantics $\text{Sem} : W \rightarrow \mathcal{P}(S)$ by

$$\text{Sem}(w_1) = \{\text{check_router}, \text{escalate_NOC}\},$$

$$\text{Sem}(w_2) = \{\text{escalate_NOC}\},$$

$$\text{Sem}(a^\star) = \{\text{escalate_NOC}\},$$

and leave w_{ill} *outside the domain*: $\text{Dom}(\text{Sem}) = \{w_1, w_2, a^\star\}$, $w_{\text{ill}} \notin \text{Dom}(\text{Sem})$ (ill-posed for the task).

Given premises $\Gamma = \{w_1, w_2\}$, the anti-philosophical closure (Def. 13.4) is

$$\begin{aligned} \text{Cn}^A(\Gamma) &= \left\{ a \in \text{Dom}(\text{Sem}) \mid \text{Sem}(w_1) \cap \text{Sem}(w_2) \subseteq \text{Sem}(a) \right\} \\ &= \left\{ a \in \text{Dom}(\text{Sem}) \mid \{\text{escalate_NOC}\} \subseteq \text{Sem}(a) \right\}. \end{aligned}$$

Hence $a^\star \in \text{Cn}^A(\Gamma)$ is an *acceptable answer*: when multiple rooms are affected and an outage is suspected, opening an incident ticket (escalation to NOC) is operationally licensed. By design, the ill-posed w_{ill} is *dissolved*: it has no semantics for this operational task and never appears among consequences.

14 Non-Philosophy

Non-Philosophy studies invariants of philosophical structures beyond decisional splits, treating them functorially to expose context-independent cores and radical immanence [191, 192].

Definition 14.1 (Philosophical structures and decisional split (Recall)). Fix a nonempty set W (*world of content*). A *philosophical structure* is a pair

$$\mathbf{P} = (\delta_{\mathbf{P}}, \text{Cn}_{\mathbf{P}}),$$

where

- $\delta_{\mathbf{P}} = (E_{\mathbf{P}}, T_{\mathbf{P}})$ with $E_{\mathbf{P}}, T_{\mathbf{P}} : W \rightarrow W$ *idempotent* endomaps satisfying $E_{\mathbf{P}} \circ T_{\mathbf{P}} = T_{\mathbf{P}} \circ E_{\mathbf{P}}$ (commuting projectors),
- $\text{Cn}_{\mathbf{P}} : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is a Tarskian closure (extensive, monotone, idempotent).

Intuitively, $\delta_{\mathbf{P}}$ encodes the *decisional split* (e.g. empirical/transcendental) used by $\mathbf{P}$.

Definition 14.2 (Radical immanence and decisional invariants). Let $\mathbf{1} := \{*\}$ and fix a *radical immanence* map $\pi : W \rightarrow \mathbf{1}$ (the unique constant map). For a philosophical structure $\mathbf{P}$, define its *invariant core*

$$\text{Inv}(\mathbf{P}) := \{x \in W \mid E_{\mathbf{P}}(x) = T_{\mathbf{P}}(x)\}.$$

A *non-philosophical observable* on $\mathbf{P}$ is any map $F_{\mathbf{P}} : \text{Inv}(\mathbf{P}) \rightarrow \mathbf{1}$ such that $F_{\mathbf{P}}|_{\text{Inv}(\mathbf{P})} = \pi|_{\text{Inv}(\mathbf{P})}$ (hence $F_{\mathbf{P}}$ ignores the split).

Definition 14.3 (Non-Philosophy). Let $\mathbb{P}$ be the class of philosophical structures and $\text{Mor}(\mathbb{P})$ the maps $h : (W, \delta_{\mathbf{P}}, \text{Cn}_{\mathbf{P}}) \rightarrow (W, \delta_{\mathbf{Q}}, \text{Cn}_{\mathbf{Q}})$ preserving projectors ($h \circ E_{\mathbf{P}} = E_{\mathbf{Q}} \circ h$, $h \circ T_{\mathbf{P}} = T_{\mathbf{Q}} \circ h$) and reflecting closure ($h[\text{Cn}_{\mathbf{P}}(X)] \subseteq \text{Cn}_{\mathbf{Q}}(h[X])$). A *Non-Philosophy* is the functor

$$\text{NP} : (\mathbb{P}, \text{Mor}(\mathbb{P})) \longrightarrow \mathbf{Set}, \quad \mathbf{P} \longmapsto \text{Inv}(\mathbf{P}),$$

together with the family $\{\pi_{\mathbf{P}} = \pi|_{\text{Inv}(\mathbf{P})} : \text{Inv}(\mathbf{P}) \rightarrow \mathbf{1}\}_{\mathbf{P} \in \mathbb{P}}$. Its *theorems* are the natural transformations built from $\pi_{\mathbf{P}}$ (i.e. statements depending only on $\text{Inv}(\mathbf{P})$ and natural in $\mathbf{P}$).

Example 14.4 (Hospital triage as Non-Philosophy). Let W be the set of clinical actions in an emergency room. Each department (cardiology, trauma, pediatrics, *etc.*) induces a *decisional split* $\delta_{\mathbf{P}} = (E_{\mathbf{P}}, T_{\mathbf{P}})$ on W : $E_{\mathbf{P}}$ keeps empirically measurable steps (vital signs, oxygen saturation), while $T_{\mathbf{P}}$ keeps steps licensed by the department's theoretical framework (protocols, specialty heuristics). The *non-philosophical invariant core*

$$\text{Inv}(\mathbf{P}) := \{x \in W \mid E_{\mathbf{P}}(x) = T_{\mathbf{P}}(x)\}$$

collects triage procedures accepted by *both* sides—e.g., the ABC rule (*airway* $\rightarrow$ *breathing* $\rightarrow$ *circulation*) and “defibrillator ready before imaging.” No matter which department's philosophy governs the case, the functor NP sends it to the same invariant practices $\text{Inv}(\mathbf{P})$, thereby capturing the *radically immanent* core of safe care that ignores the split.

15 Critical Philosophy

Critical Philosophy formalizes Kant’s project: restricting valid knowledge to structured appearances, while leaving noumena unconstrained, ensuring metaphysics remains non-dogmatic and bounded [193–196]. Related concepts include *Critical Thinking*, which is frequently applied in fields such as social sciences and business [197–199].

Definition 15.1 (Critical Philosophy). A *Critical Philosophy* is a tuple

$$\mathbf{CP} = (\mathcal{L}_{\text{crit}}, \text{Ax}_{\text{tr}}, \text{Mod}_{\text{crit}}, \models_{\text{crit}}, \text{Cn}_{\text{crit}})$$

satisfying:

1. **Language.** $\mathcal{L}_{\text{crit}}$ is a many-sorted first-order language with at least the sorts $\mathbb{E}$ (appearances) and $\mathbb{N}$ (noumena) and designated *form/category* symbols on $\mathbb{E}$ (e.g. $\text{tm} : \mathbb{E} \rightarrow \mathbb{T}$, $\text{sp} : \mathbb{E} \rightarrow \mathbb{S}$, and a causal relation $\text{cause} \subseteq \mathbb{E} \times \mathbb{E}$). A sentence is *empirical* if it uses no $\mathbb{N}$ -symbols.
2. **Transcendental axioms.** $\text{Ax}_{\text{tr}} \subseteq \text{Sent}(\mathcal{L}_{\text{crit}})$ is a fixed, consistent set of axioms *only about* $\mathbb{E}$ that express the a priori forms and categories (e.g. linear order on time, spatial form, and time-respecting, acyclic causality).
3. **Critical models.**

$$\text{Mod}_{\text{crit}} := \left\{ \mathcal{M} \mid \mathcal{M} \models \text{Ax}_{\text{tr}} \text{ on the } \mathbb{E}\text{-part, and the } \mathbb{N}\text{-part is arbitrary (nonempty)} \right\}.$$

Thus the noumenal domain and all $\mathbb{N}$ -symbols are left unconstrained beyond nonemptiness.

4. **Validity and consequence.** For $\Gamma \subseteq \text{Sent}(\mathcal{L}_{\text{crit}})$ and φ ,

$$\Gamma \models_{\text{crit}} \varphi \iff \forall \mathcal{M} \in \text{Mod}_{\text{crit}} (\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi),$$

and $\text{Cn}_{\text{crit}}(\Gamma) := \{\varphi : \Gamma \models_{\text{crit}} \varphi\}$. Then Cn_{crit} is extensive, monotone, and idempotent.

5. **Boundary (noumenal non-assertion).** If Γ is empirical and $\Gamma \models_{\text{crit}} \varphi$ while φ mentions $\mathbb{N}$, then φ is invariant under arbitrary changes of the $\mathbb{N}$ -part (hence carries no informative, assertoric content about noumena).

Example 15.2 (Black-box AI auditing as Critical Philosophy). A regulator audits a proprietary credit-scoring model without access to its internals.

- **Appearances $\mathbb{E}$:** logged inputs, outputs, timestamps, and test scenarios; observed error and disparity rates.
- **Noumena $\mathbb{N}$:** hidden architecture, weights, training corpus—kept arbitrary and unconstrained.
- **Transcendental axioms Ax_{tr} :** time is linearly ordered (no backward influence), causality on $\mathbb{E}$ is acyclic (test setup $\rightarrow$ output), and repeated trials under fixed conditions yield stable frequencies.
- **Critical consequence:** Valid audit claims (e.g., “disparate impact $< \epsilon$ ” or “AUC ≥ 0.82 on cohort C ”) are stated *only* about $\mathbb{E}$ and remain true under any change to $\mathbb{N}$ that leaves appearances unchanged.
- **Boundary principle:** No assertoric claims are made about internal mechanisms; metaphysical speculation about “true fairness essence” is excluded as non-empirical.

Thus knowledge is restricted to structured appearances, while noumenal details are left open—exactly the stance of Critical Philosophy in a real compliance workflow.

16 Partial Philosophy

Partial Philosophy provides partial semantics: only some propositions are interpreted, closure operates within scope, leaving others uninterpreted.

Definition 16.1 (Partial Philosophy). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ an address map. Let $(L, \leq)$ be a poset of semantic values and

$$\text{Sem} : A \rightarrow L$$

a *partial* semantics with domain $\text{Dom}(\text{Sem}) \subseteq A$. Fix a nonempty *scope* $Q_0 \subseteq Q$ and require

$$\text{adr}[\text{Dom}(\text{Sem})] \subseteq Q_0.$$

A *Partial Philosophy* is a tuple

$$\mathbf{PP} := (A, Q, \text{adr}, L, \text{Sem}, Q_0, \text{Cn}^{\text{part}}),$$

where $\text{Cn}^{\text{part}} : \mathcal{P}(\text{Dom}(\text{Sem})) \rightarrow \mathcal{P}(\text{Dom}(\text{Sem}))$ is a Tarskian closure operator (extensive, monotone, idempotent) that is sound with respect to Sem (on $\text{Dom}(\text{Sem})$).

Given premises $\Gamma \subseteq \text{Dom}(\text{Sem})$ and a question $q \in Q_0$, the set of in-scope answers is

$$\text{Ans}^{\text{part}}(q \mid \Gamma) := \text{Cn}^{\text{part}}(\Gamma) \cap \text{adr}^{-1}(\{q\}).$$

Statements outside $\text{Dom}(\text{Sem})$ are *out of scope* (left uninterpreted).

Example 16.2 (AI procurement with a limited, in-scope semantics). A city evaluates a face-recognition system under a *Partial Philosophy* that only interprets safety-critical questions.

Let

$$Q = \{\text{privacy, bias, safety, accuracy, budget, deploy?}\}, \quad Q_0 = \{\text{privacy, bias, safety}\}.$$

Let A contain audit claims such as

$$a_1 : \text{“privacy risk} \geq \text{medium”} \quad (\text{adr}(a_1) = \text{privacy}), \quad a_2 : \text{“FNR gap} \leq 3\%” \quad (\text{adr}(a_2) = \text{bias}),$$

$$a_3 : \text{“incident rate} < 10^{-5}/\text{hr”} \quad (\text{adr}(a_3) = \text{safety}), \quad d : \text{“deploy the system”} \quad (\text{adr}(d) = \text{deploy?}).$$

Take $L = \{\text{low} \leq \text{medium} \leq \text{high}\}$. Define a *partial* semantics $\text{Sem} : A \rightarrow L$ by mapping audit-supported claims a_1, a_2, a_3 to their measured levels in L , while leaving $\text{Sem}(d)$ and any $\text{adr} \notin Q_0$ *undefined*. The closure Cn^{part} derives in-scope consequences (e.g. “privacy risk is at least medium” $\Rightarrow$ “privacy mitigations required”), but gives no meaning to out-of-scope statements like deploy? .

Thus, acceptable answers are produced for $q \in Q_0$ (privacy/bias/safety) from the audits, while policy/financial decisions remain *uninterpreted* (out of scope), illustrating a real-life partial, safety-focused evaluation.

17 Dual-Structure and Dual-philosophy

A *Dual Structure* is obtained by applying a contravariant dualization functor, reversing operations or relations while preserving consistency. A *Dual-Philosophy* pairs a philosophical structure with its dual, ensuring entailments in one correspond anti-isomorphically to entailments in the other.

Definition 17.1 (Signature, structures, morphisms). Fix a single-sorted finitary signature

$$\Sigma = (\text{Func}, \text{Rel}, \text{ar}_{\text{Func}}, \text{ar}_{\text{Rel}}).$$

A Σ -structure is a tuple

$$\mathbf{A} = (A, (f^{\mathbf{A}})_{f \in \text{Func}}, (R^{\mathbf{A}})_{R \in \text{Rel}}),$$

with nonempty carrier A , each $f^{\mathbf{A}} : A^m \rightarrow A$ of arity $m = \text{ar}(f)$, and each $R^{\mathbf{A}} \subseteq A^r$ of arity $r = \text{ar}(R)$. A homomorphism $h : \mathbf{A} \rightarrow \mathbf{B}$ is a map $h : A \rightarrow B$ preserving all function and relation symbols in the usual way. Let Str_{Σ} denote the category of Σ -structures and homomorphisms.

Definition 17.2 (Dualization functor and dual structure). A *dualization* on Str_Σ is a contravariant endofunctor

$$D : \text{Str}_\Sigma^{\text{op}} \longrightarrow \text{Str}_\Sigma$$

together with a natural isomorphism (the *involution witness*)

$$\varepsilon : \text{Id}_{\text{Str}_\Sigma} \xrightarrow{\cong} D \circ D.$$

For any Σ -structure $\mathbf{A}$, its *dual structure* is

$$\mathbf{A}^* := D(\mathbf{A}).$$

For a homomorphism $h : \mathbf{A} \rightarrow \mathbf{B}$, the *dual morphism* is $h^* := D(h) : \mathbf{B}^* \rightarrow \mathbf{A}^*$.

Remark 17.3 (How D acts on symbols). Concrete dualizations are specified by uniform *recipes* on symbol interpretations. Typical examples (each gives a functor D):

- **Posets:** $\Sigma = \{\leq\}$, $R^{\mathbf{A}} = \leq^{\mathbf{A}} \subseteq A^2$; define $\mathbf{A}^*$ by reversing the order: $x \leq^{\mathbf{A}^*} y \iff y \leq^{\mathbf{A}} x$.
- **Directed graphs:** $\Sigma = \{E\}$; set $E^{\mathbf{A}^*} = \{(v, u) : (u, v) \in E^{\mathbf{A}}\}$.
- **Incidence dual of hypergraphs:** swap vertex/edge incidence via transposition of the incidence relation.
- **Topological/vector duals (categorical view):** interpret D as the appropriate contravariant *Hom*-functor (e.g. $V \mapsto V^* = \text{Hom}(V, \mathbb{K})$), with ε given by the canonical bidual map when reflexivity holds.

In each case, D is isomorphism-invariant and functorial.

Definition 17.4 (Dual-Philosophy). Let Σ be a fixed single-sorted finitary signature and let Str_Σ be the category of Σ -structures and homomorphisms (cf. Theorem 17.1). A *dualization* is a contravariant endofunctor

$$D : \text{Str}_\Sigma^{\text{op}} \longrightarrow \text{Str}_\Sigma$$

equipped with a natural isomorphism (involution witness) $\varepsilon : \text{Id}_{\text{Str}_\Sigma} \Rightarrow D \circ D$ (cf. Theorem 17.2).

A *Dual-Philosophy* is a pair $(\mathbf{P}, D)$ where:

- $\mathbf{P} \in \text{Str}_\Sigma$ is a *philosophical structure*, i.e. its carrier interprets the basic ingredients of a philosophy (propositions A , questions Q , addressing adr , and a consequence operator Cn , encoded inside Σ);
- D is a dualization on Str_Σ .

Its *dual philosophy* is $\mathbf{P}^* := D(\mathbf{P})$. If $h : \mathbf{P} \rightarrow \mathbf{P}'$ is a morphism of philosophies, the *dual morphism* is $h^* := D(h) : \mathbf{P}'^* \rightarrow \mathbf{P}^*$.

Definition 17.5 (Dual entailment). Write the (object-level) consequence of $\mathbf{P}$ as a closure operator $\text{Cn}_{\mathbf{P}} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ on the carrier A of $\mathbf{P}$ (so $\Gamma \vdash_{\mathbf{P}} a$ iff $a \in \text{Cn}_{\mathbf{P}}(\Gamma)$). Let $\iota_{\mathbf{P}} : A \xrightarrow{\cong} A^{**}$ be the component of ε (after forgetting structure). Define the *dual entailment* $\vdash_{\mathbf{P}^*}$ on the carrier A^* of $\mathbf{P}^*$ by

$$\Gamma^* \vdash_{\mathbf{P}^*} a^* \iff \iota_{\mathbf{P}}^{-1}((a^*)^*) \in \text{Cn}_{\mathbf{P}}(\iota_{\mathbf{P}}^{-1}((\Gamma^*)^*)),$$

i.e. a is entailed by Γ in $\mathbf{P}$ precisely when the *dualized* statement a^* is entailed by the *dualized* premises Γ^* in $\mathbf{P}^*$ (identifying $A \cong A^{**}$ via $\iota_{\mathbf{P}}$).

Remark 17.6 (Symbol action). Operationally, D is specified by uniform recipes on the symbols of Σ (order reversal for posets, edge reversal for digraphs, incidence transpose for hypergraphs, contravariant *Hom* for linear spaces, etc.). This induces the corresponding transformation of Cn encoded in the structure, hence the dual entailment in Theorem 17.5.

Proposition 17.7 (Anti-isomorphism of entailment preorders). *Let $(\mathbf{P}, D)$ be a Dual-Philosophy. The map $a \mapsto a^*$ (with identification $A \cong A^{**}$) yields an order anti-isomorphism between the entailment preorders:*

$$\Gamma \vdash_{\mathbf{P}} a \iff \Gamma^* \vdash_{\mathbf{P}^*} a^*.$$

Proof. By Theorem 17.5 and naturality of ε , $\Gamma \vdash_{\mathbf{P}} a$ iff $a \in \text{Cn}_{\mathbf{P}}(\Gamma)$ iff, after applying D twice and using ε , the corresponding dual statements satisfy the displayed equivalence. $\square$

18 Iterated Dual-Structure and Iterated Dual-philosophy

An Iterated Dual Structure applies dualization repeatedly, forming a tower where even levels recover the original and odd levels its dual. Iterated Dual-Philosophy extends this principle to philosophical structures, ensuring entailments alternate between philosophy and its dual across successive iterations.

Definition 18.1 (Dualization). Fix a single-sorted finitary signature Σ and let Str_Σ be the category of Σ -structures and homomorphisms. A *dualization* is a contravariant endofunctor

$$D : \text{Str}_\Sigma^{\text{op}} \longrightarrow \text{Str}_\Sigma$$

together with a natural isomorphism (the *involution witness*)

$$\varepsilon : \text{Id}_{\text{Str}_\Sigma} \xrightarrow{\cong} D \circ D.$$

For $A \in \text{Str}_\Sigma$, write $A^* := D(A)$.

Definition 18.2 (Iterated duals and the dual tower). Let (D, ε) be a dualization. Define $D^0 := \text{Id}$ and $D^{n+1} := D \circ D^n$ for $n \geq 0$. For a structure $P \in \text{Str}_\Sigma$ set

$$P^{*n} := D^n(P) \quad (n \in \mathbb{N}).$$

The sequence

$$\text{Tower}(P, D) := (P^{*0} = P, P^{*1}, P^{*2}, \dots)$$

is the *iterated dual tower*. The components of ε yield canonical isomorphisms

$$\varepsilon_{P^{*n}} : P^{*n} \xrightarrow{\cong} P^{*(n+2)} \quad (n \geq 0).$$

Definition 18.3 (Iterated Dual-Philosophy). Assume Σ encodes a *philosophical structure* $P = (A, \dots, \text{Cn}_P)$ whose carrier A contains propositions and whose (consequence/closure) operator $\text{Cn}_P : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is extensive, monotone, and idempotent. An *Iterated Dual-Philosophy* is the pair (P, D) together with its dual tower from Theorem 18.2. For each n , write

$$\text{Cn}_n := \text{Cn}_{P^{*n}} : \mathcal{P}(A^{*n}) \longrightarrow \mathcal{P}(A^{*n}), \quad \Gamma \vdash_n a \iff a \in \text{Cn}_n(\Gamma).$$

Theorem 18.4 (Periodicity). For every $P \in \text{Str}_\Sigma$ and $n \in \mathbb{N}$ there are natural isomorphisms

$$P^{**} \cong P, \quad P^{*(n+2)} \cong P^{*n}.$$

Hence all even levels are (naturally) isomorphic to P and all odd levels to P^* .

Proof. The first is $\varepsilon_P : P \xrightarrow{\cong} D^2(P)$. For the second, apply ε at level P^{*n} : $\varepsilon_{P^{*n}} : P^{*n} \xrightarrow{\cong} P^{*(n+2)}$. $\square$

Proposition 18.5 (Adjacency law (dual entailment)). Let $u_n : A^{*n} \xrightarrow{\cong} A^{*(n+2)}$ be the underlying carrier isomorphism of $\varepsilon_{P^{*n}}$. For all $\Gamma \subseteq A^{*n}$ and $a \in A^{*n}$,

$$\Gamma \vdash_n a \iff D(\Gamma) \vdash_{n+1} D(a),$$

where $D(\Gamma) := \{D(x) : x \in \Gamma\} \subseteq A^{*(n+1)}$ and the identification $A^{*n} \cong A^{*(n+2)}$ is taken via u_n .

Proof. By functoriality, D transports the structure interpreting Cn_n to that interpreting Cn_{n+1} . Using u_n to shuttle between n and $n+2$, one obtains the displayed equivalence directly from the definitions of $\vdash_n$ and $\vdash_{n+1}$. $\square$

19 Multipolar-Structure and Multipolar-Philosophy

Multipolar-Philosophy evaluates propositions across multiple poles or viewpoints, aggregating weighted perspectives to derive balanced conclusions without privileging a single standpoint.

Definition 19.1 (Truth-degree algebra (Recall)). Let $\mathbf{L} = (L, \leq, T, S, 0, 1)$ where $(L, \leq)$ is a complete lattice with bottom 0 and top 1, $T : L^2 \rightarrow L$ is a left-continuous t-norm (commutative, associative, monotone, $T(x, 1) = x$), and $S : L^2 \rightarrow L$ is its t-conorm (on $[0, 1]$, $S = \max$). For $s \geq 1$ write $T_s(\alpha_1, \dots, \alpha_s)$ for the iterated t-norm.

Definition 19.2 (Truth-degree setting (Recall)). Let $\mathbf{L} = (L, \leq, T, S, 0, 1)$ be a truth-degree algebra, where $(L, \leq)$ is a complete lattice with bottom 0 and top 1, $T : L^2 \rightarrow L$ is a (left-continuous) t-norm, and $S : L^2 \rightarrow L$ its t-conorm. For $s \geq 1$ write $T_s(\alpha_1, \dots, \alpha_s) := T(\alpha_1, T_{s-1}(\alpha_2, \dots, \alpha_s))$ (with $T_1(\alpha) = \alpha$). For $u, v \in L^\Pi$ (component vectors indexed by a finite nonempty set Π of *poles*), write $u \leq_\Pi v \iff (\forall \pi \in \Pi) u_\pi \leq v_\pi$.

Definition 19.3 (Multipolar-Structure). A *Multipolar-Structure over $\mathbf{L}$* is a tuple

$$\mathbf{MS} = (H, \{\#^{(m)} : H^m \rightarrow H\}_{m \in \mathcal{I}}, \Pi, v, \text{Agg}),$$

consisting of:

1. a nonempty carrier set H and a family of basic operations $\#^{(m)}$ for arities $m \in \mathcal{I} \subseteq \mathbb{N}_{\geq 1}$;
2. a finite, nonempty set of *poles* Π (viewpoints, facets);
3. a *polar valuation* $v : H \rightarrow L^\Pi$, assigning to each $x \in H$ its component degrees $v(x)_\pi \in L$ at each pole $\pi \in \Pi$;
4. a monotone *aggregation* $\text{Agg} : L^\Pi \rightarrow L$ (e.g. weighted supremum or mean) used to obtain a single collapsed score $\bar{v} := \text{Agg} \circ v$ when desired.

These data satisfy the *polar compositionality* axiom: for every $m \in \mathcal{I}$ and $x_1, \dots, x_m \in H$,

$$v(\#^{(m)}(x_1, \dots, x_m)) \geq_\Pi F^{(m)}(v(x_1), \dots, v(x_m)), \quad \text{where} \quad (F^{(m)}(u^{(1)}, \dots, u^{(m)}))_\pi := T_s(u_\pi^{(1)}, \dots, u_\pi^{(m)}).$$

(Thus each pole evaluates the output at least as strongly as the T -combination of the inputs' polewise degrees; $s = m$ above. Additional algebraic laws on $\#^{(m)}$ (associativity, identities, etc.) may be imposed and are reflected componentwise via v .)

Definition 19.4 (Multipolar-Philosophy). Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\text{adr} : A \rightarrow Q$ an addressing map. Let Π be a finite nonempty set of *poles* (e.g. viewpoints, stances, semantic facets). A *multipolar state* is a map $\mu \in (L^\Pi)^A$; we write $\mu(a)_\pi \in L$ for the degree of $a \in A$ at pole $\pi \in \Pi$.

Fix nonempty sets of *methods* M and *contexts* R . For each $(m, r) \in M \times R$ let

$$\mathcal{R}_{m,r} \subseteq \bigcup_{s \geq 1} (A^s \times A) \times L^\Pi$$

be a (possibly infinite) family of *multipolar rules*. An element $\rho \in \mathcal{R}_{m,r}$ is written

$$\rho : (a_1, \dots, a_s) \Rightarrow b \quad \text{with strength vector } \lambda_\rho \in L^\Pi,$$

meaning that, at pole π , the rule contributes at least $T(\lambda_\rho(\pi), T_s(\mu(a_1)_\pi, \dots, \mu(a_s)_\pi))$.

To model cross-pole interaction, fix a monotone *polar mixing* map $K_{m,r} : L^\Pi \rightarrow L^\Pi$ of the form

$$(K_{m,r}(x))_\pi := \bigvee_{\kappa \in \Pi} T(W_{m,r}(\pi, \kappa), x_\kappa),$$

where $W_{m,r} : \Pi \times \Pi \rightarrow L$ is a given weight matrix (e.g. on $[0, 1]$, entries in $[0, 1]$).

One-step derivation $\text{Der}_{m,r}^{\text{MP}} : (L^\Pi)^A \rightarrow (L^\Pi)^A$ is defined componentwise by

$$\text{Der}_{m,r}^{\text{MP}}(\mu)(b) := K_{m,r} \left(\left[\mu(b)_\pi \ S \ \bigvee_{\rho \in \mathcal{R}_{m,r}} \bigvee_{\substack{(a_1, \dots, a_s) \\ \rho:(a_1, \dots, a_s) \Rightarrow b}} T(\lambda_\rho(\pi), T_s(\mu(a_1)_\pi, \dots, \mu(a_s)_\pi)) \right]_{\pi \in \Pi} \right).$$

Given an initial state μ_0 (e.g. premises $\Gamma \subseteq A$ encoded by $\mu_0(a)_\pi = 1$ if $a \in \Gamma$ else 0), define $\mu_{t+1} := \text{Der}_{m,r}^{\text{MP}}(\mu_t)$ and the *multipolar closure*

$$\text{Cn}_{m,r}^{\text{MP}}(\mu_0) := \bigvee_{t \geq 0} \mu_t \in (L^\Pi)^A \quad (\text{pointwise supremum in } L).$$

For $q \in Q$, the *acceptable answers at pole π* are

$$\text{Ans}^{\text{MP}}(q \mid \mu_0; m, r)(a)_\pi := \begin{cases} \text{Cn}_{m,r}^{\text{MP}}(\mu_0)(a)_\pi, & \text{adr}(a) = q, \\ 0, & \text{otherwise.} \end{cases}$$

The tuple

$$\mathbf{MP} := (A, Q, \text{adr}, \Pi, M, R, \mathbf{L}, \{W_{m,r}\}, \{\mathcal{R}_{m,r}\})$$

is a *Multipolar-Philosophy*.

Proposition 19.5 (Tarskian properties). *For fixed (m, r) , $\text{Der}_{m,r}^{\text{MP}}$ is extensive and monotone; hence $\text{Cn}_{m,r}^{\text{MP}}$ is an extensive, monotone, idempotent operator on $(L^\Pi)^A$ (least fixed point above μ_0).*

Proof. Extensivity: for each b, π the inner S keeps $\mu(b)_\pi \leq \text{Der}_{m,r}^{\text{MP}}(\mu)(b)_\pi$, and $K_{m,r}$ is monotone with $W_{m,r}(\pi, \pi) \geq 0$, so the output is $\geq \mu(b)_\pi$. Monotonicity: all constructions (T, S , suprema, $K_{m,r}$) are monotone in μ . Idempotence then follows by standard Knaster–Tarski arguments on the complete lattice $((L^\Pi)^A, \leq)$. $\square$

20 Pseudophilosophy

Pseudophilosophy imitates philosophy’s structure without rigor: speculative, dogmatic, unsound, trivial, lacking coherent vocabulary, inference, or openness (cf. [200–202]).

Definition 20.1 (Pseudophilosophy). A discourse is encoded by a tuple

$$\Pi = (\mathcal{L}_\Sigma, \text{Ax}, \vdash, \mathcal{M}, \models),$$

where $\mathcal{L}_\Sigma$ is a first-order language over a finitary signature Σ , Ax a set of axioms, $\vdash$ a finitary proof system, $\mathcal{M}$ a class of Σ -structures, and $\models$ the satisfaction relation. Write $\text{Th}_\Pi := \{\varphi : \text{Ax} \vdash \varphi\}$.

We call Π *pseudophilosophical* iff it fails *any* of the following minimal adequacy clauses:

- (S1) **Well-defined vocabulary.** Every nonlogical symbol has either a uniform interpretation in each $M \in \mathcal{M}$ or an eliminable definition; the definitional dependency graph is well-founded (no circular/infinitary dependence).
- (S2) **Sound inference.** If $\text{Ax} \vdash \varphi$ then $M \models \varphi$ for all $M \in \mathcal{M}$.
- (S3) **Nontriviality / openness.** $\text{Th}_\Pi \neq \text{Sent}(\mathcal{L}_\Sigma)$, and for every $\varphi \notin \text{Th}_\Pi$ there exists $M \in \mathcal{M}$ with $M \models \varphi$.

Equivalently, Π is pseudophilosophical when it exhibits (i) *ungrounded vocabulary* (obscure or non-eliminable primitives), (ii) *inferential defect* (unsound rules), or (iii) *dogmatic trivialization* (immunizing schemas that force every sentence true).

Example 20.2 (Real-life pseudophilosophy: a corporate “method”). A consultancy promotes a “Quantum Synergy Decision Protocol” for management. It deploys:

1. **Ungrounded vocabulary (fails S1):** core terms like “vibrational synergy” and “quantum insight” lack precise, uniform definitions and are mutually circular.

-
2. **Unsound inference (fails S2):** success claims rest on testimonials and post hoc correlations; counterexamples are dismissed as “context misreadings” rather than addressed logically.
 3. **Dogmatic trivialization (fails S3):** an axiom states “the protocol never fails; apparent failure means improper mindset,” rendering every outcome a confirmation and blocking falsification.

Managers are urged to adopt costly trainings and reorganizations, yet no testable predictions, clear semantics, or valid evaluation criteria are provided—mimicking philosophical structure without rigor.

21 Many Other Structures

Besides the structures introduced so far, many other types of structures have also been studied. Below, we provide definitions for several such structures.

Definition 21.1 (Tree Structure). (cf. [203, 204]) Let T be a nonempty finite set, and let $\leq$ be a binary relation on T . We say that $(T, \leq)$ is a *Tree Structure* (more precisely, a finite rooted tree) if and only if the following conditions hold:

- (i) $\leq$ is a partial order on T . In other words, for all $x, y, z \in T$:
 - $x \leq x$ (reflexivity),
 - If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),
 - If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).
- (ii) There exists a unique minimal element $r \in T$ (called the *root*) such that

$$\forall x \in T, r \leq x.$$

- (iii) For every $x \in T$, the set

$$\{y \in T \mid y \leq x\}$$

is totally ordered by $\leq$. That is, if $y_1, y_2 \in T$ and $y_1, y_2 \leq x$, then either $y_1 \leq y_2$ or $y_2 \leq y_1$.

In this situation, we call T a finite rooted tree under the order $\leq$, and refer to $(T, \leq)$ as a *Tree Structure*.

Example 21.2 (Tree Structure — corporate reporting lines). Let T be the set of roles in a company (CEO, VP, Director, Manager, . . .). Define $x \leq y$ iff role x is on the chain of command from the CEO to role y (ancestor in the org chart). Then $(T, \leq)$ is a rooted tree with the CEO as the unique root; every node’s ancestors form a total order (CEO $\leq$ VP $\leq$ Director $\leq$. . .).

Definition 21.3 (Forest Structure). (cf. [203, 204]) Let F be a nonempty finite set, and let $\leq$ be a binary relation on F . We say that $(F, \leq)$ is a *Forest Structure* if and only if the following conditions hold:

- (i) $\leq$ is a partial order on F . In other words, for all $x, y, z \in F$:
 - $x \leq x$ (reflexivity),
 - If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),
 - If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).
- (ii) For each $x \in F$, the *principal ideal*

$$\downarrow x := \{y \in F \mid y \leq x\}$$

is totally ordered by $\leq$. That is, if $y_1, y_2 \in \downarrow x$, then either $y_1 \leq y_2$ or $y_2 \leq y_1$.

Equivalently, $(F, \leq)$ is a finite poset whose Hasse diagram is a disjoint union of rooted trees; each connected component is a rooted tree. We call $(F, \leq)$ a *Forest Structure*.

Example 21.4 (Forest Structure — university schools and departments). Let F be the set of academic units in a university. Each school (Engineering, Medicine, Arts) forms its own rooted tree: Dean $\leq$ Department Chair $\leq$ Program $\leq$ Lab. The poset $(F, \leq)$ is therefore a disjoint union of such trees—a forest—since units in different schools are unrelated by $\leq$.

Definition 21.5 (Curried Structure). [15, 104] Let H be a nonempty set. A *Curried Structure* on H is a pair

$$(H, \varphi), \quad \varphi: H \rightarrow (H \rightarrow H),$$

where $\varphi(x)$ is a function $H \rightarrow H$. Equivalently, $\varphi: H \rightarrow H \rightarrow H$.

Example 21.6 (Curried Structure — reusable “prefixer” for subject lines). (cf. [15, 104]) Let H be the set of email subject strings. Define $\varphi: H \rightarrow (H \rightarrow H)$ by $\varphi(t)(s) := ts$ (concatenation). Fixing a tag $t = \text{“[URGENT]”}$ yields the function $\varphi(t): H \rightarrow H$ that prefixes any subject s with [URGENT] , modeling a real-world reusable action “add this tag to any subject”.

Definition 21.7 (Curried k -ary Structure). [15] Let H be a nonempty set (the *carrier*) and $I \subseteq \mathbb{Z}_{>0}$ a family of arities. A *curried k -ary structure* of signature I on H consists of maps

$$\{ f^{(m)}: H \rightarrow \underbrace{H \rightarrow \cdots \rightarrow H}_{m \text{ times}} \}_{m \in I},$$

where each $f^{(m)}$ is termed the *curried m -ary operation*, sometimes notated

$$f^{(m)}: H \rightarrow H \rightarrow \cdots \rightarrow H \quad (m \text{ arrows}).$$

Example 21.8 (Curried k -ary Structure — stepwise path builder (here $k = 3$)). Let H be the set of file path strings. Define a curried ternary operation $f^{(3)}: H \rightarrow (H \rightarrow (H \rightarrow H))$ by

$$f^{(3)}(r)(d)(p) := r/d/p,$$

the usual path join. Applying arguments one at a time mirrors practice: first choose a root drive r , then a division d , then a project p , producing the final path in H . (Equivalently, the uncurried $g: H^3 \rightarrow H$, $g(r, d, p) = r/d/p$, factors as $f^{(3)}$.)

Definition 21.9 (U -Structure). Let $C = (H, \{\#^{(m)}\})$ be a Classical Structure, with carrier H . A *U -Structure* C^U is the pair

$$C^U = \left(H, \mu: H \rightarrow \text{Dom}(U), \{\#^{(m)}\} \right),$$

where

$$\mu(x) \in \text{Dom}(U) \quad \text{is the } U\text{-membership degree of } x \in H.$$

The underlying classical operations $\{\#^{(m)}\}$ remain those of C , but each element of H carries a U -membership label $\mu(x)$. In effect, C^U is a *fuzzy- or neutrosophic-enrichment* of the classical structure C .

Example 21.10 (U -Structure — risk-labeled tasks). Let H be the set of open work items in a team’s backlog and take the classical structure $C = (H, \emptyset)$ (no operations needed). Define a membership map $\mu: H \rightarrow [0, 1]$ where $\mu(x)$ is the normalized *risk/urgency* of item x ($0 = \text{negligible}$, $1 = \text{critical}$). Then $C^U = (H, \mu, \emptyset)$ is a U -Structure used in practice for triage: sorting, batching, or alerting policies depend on $\mu(x)$ while the underlying items H remain unchanged.

Definition 21.11 (Path-Structure). Let $k \in \mathbb{Z}_{\geq 0}$. A *Path-Structure of length k* is the finite, simple, undirected graph

$$P_k = (V, E), \quad V = \{v_0, v_1, \dots, v_k\}, \quad E = \{\{v_i, v_{i+1}\} : 0 \leq i < k\}.$$

Thus P_k has $k + 1$ vertices and k edges, the endpoints v_0, v_k have degree 1 (if $k \geq 1$), and all intermediate vertices $v_1, \dots, v_{k-1}$ have degree 2. (An oriented version uses arcs (v_i, v_{i+1}) for $0 \leq i < k$.) A *path-structure* in a graph G is any subgraph of G isomorphic to some P_k .

Example 21.12 (Path-Structure — linear approval workflow). Consider an expense request that must pass through four roles in order: Employee $\rightarrow$ Manager $\rightarrow$ Finance $\rightarrow$ CFO. This forms a path P_3 with vertices $v_0 = \text{Employee}$, $v_1 = \text{Manager}$, $v_2 = \text{Finance}$, $v_3 = \text{CFO}$ and edges $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$. The request advances along this path until final approval.

Definition 21.13 (Cycle-Structure). Let $n \in \mathbb{Z}_{\geq 3}$. A *Cycle-Structure of length n* is the finite, simple, undirected graph

$$C_n = (V, E), \quad V = \{v_0, v_1, \dots, v_{n-1}\}, \quad E = \{\{v_i, v_{i+1 \bmod n}\} : 0 \leq i < n\}.$$

Thus C_n has n vertices and n edges, and every vertex has degree 2. (An oriented version uses arcs $(v_i, v_{i+1 \bmod n})$ for $0 \leq i < n$.) A *cycle-structure in a graph G* is any subgraph of G isomorphic to some C_n .

Example 21.14 (Cycle-Structure — rotating on-call schedule). Let $V = \{e_0, \dots, e_{n-1}\}$ be engineers on a weekly on-call rotation. Connect each e_i to $e_{(i+1) \bmod n}$; the subgraph is a cycle C_n . After each week the duty “hands off” along an edge, returning to e_0 after n weeks, modeling the recurring schedule.

Definition 21.15 (Mixed MetaStructure). Fix a nonempty set T of *types*. For each $a \in \mathsf{T}$, let $\Sigma^{(a)} = (\text{Sort}^{(a)}, \text{Func}^{(a)}, \text{Rel}^{(a)}, \text{ar})$ be a *multi-sorted* finitary signature. Write $\text{Str}_{\Sigma^{(a)}}$ for the class of all (multi-sorted) $\Sigma^{(a)}$ -structures, and $U_a \subseteq \text{Str}_{\Sigma^{(a)}}$ for a fixed, nonempty, isomorphism-closed collection of such structures.

A *Mixed MetaStructure* over the typed family $\{\Sigma^{(a)}\}_{a \in \mathsf{T}}$ is a pair

$$\mathbb{M}_{\text{mix}} := \left((U_a)_{a \in \mathsf{T}}, (\Phi_\ell)_{\ell \in \Lambda} \right),$$

where each *mixed meta-operation* Φ_ℓ has a typed *profile* $\mathbf{a}_\ell = (a_1, \dots, a_k | b) \in \mathsf{T}^k \times \mathsf{T}$ and is a mapping

$$\Phi_\ell : U_{a_1} \times \dots \times U_{a_k} \longrightarrow U_b,$$

specified by uniform *recipes* on carriers and symbol interpretations of the *target* signature $\Sigma^{(b)}$:

- (i) For each sort $s \in \text{Sort}^{(b)}$, a carrier constructor $\Gamma_\ell^{(s)}(\mathbf{C}_1, \dots, \mathbf{C}_k) = H_s^{(b)}$ built functorially from the input carriers;
- (ii) For each function symbol $f \in \text{Func}^{(b)}$ of arity $\vec{s} \rightarrow s$, a recipe

$$\Lambda_\ell^f((f_i^{C_i})_{i=1}^k) : (H_{\vec{s}}^{(b)}) \longrightarrow H_s^{(b)}$$

yielding the interpretation $f^{\Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_k)}$;

- (iii) For each relation symbol $R \in \text{Rel}^{(b)}$ of arity $\vec{s}$, a recipe $\Xi_\ell^R((R_i^{C_i})_{i=1}^k) \subseteq H_{\vec{s}}^{(b)}$ yielding $R^{\Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_k)}$.

These recipes depend only on the profile $\mathbf{a}_\ell$ and the symbols of $\Sigma^{(b)}$ (not on representatives).

Example 21.16 (Mixed data integration for “Customer 360”). Let the type set be $\mathsf{T} = \{\text{RDB}, \text{Graph}, \text{Stream}, \text{Dash}\}$. Here U_{RDB} are relational customer/order schemas, U_{Graph} are product-relation or social graphs, U_{Stream} are click/transaction event streams, and U_{Dash} are dashboard/report schemas. A mixed meta-operation

$$\Phi_{\text{unify}} : U_{\text{RDB}} \times U_{\text{Graph}} \times U_{\text{Stream}} \longrightarrow U_{\text{Dash}}$$

constructs a denormalized star schema with KPIs (retention, LTV, churn signals) by joining tables, projecting graph features (e.g. centrality), and aggregating streams (windowed counts), yielding a single analytics structure for business users.

Example 21.17 (Smart factory maintenance planning). Let $\mathsf{T} = \{\text{BOM}, \text{Sensor}, \text{Sched}, \text{Plan}\}$. Here U_{BOM} are bill-of-materials trees, U_{Sensor} are time-series/event schemas from machines, U_{Sched} are shift/availability schedules, and U_{Plan} are maintenance-plan structures (tasks with precedence/resources). Define

$$\Phi_{\text{plan}} : U_{\text{BOM}} \times U_{\text{Sensor}} \times U_{\text{Sched}} \longrightarrow U_{\text{Plan}}$$

that fuses failure signals (from sensors) with component hierarchies (BOM) and staff availability (schedule) to produce a feasible, resource-aware maintenance plan.

Example 21.18 (Hospital triage decision support). Let $\mathsf{T} = \{\text{EHR}, \text{Ontology}, \text{Image}, \text{Rule}\}$. Here U_{EHR} are relational encounter/lab/medication schemas, U_{Ontology} are diagnosis/procedure graphs (e.g. ICD/SNOMED), U_{Image} are report trees (findings/sections), and U_{Rule} are clinical decision rules. A mixed meta-operation

$$\Phi_{\text{triage}} : U_{\text{EHR}} \times U_{\text{Ontology}} \times U_{\text{Image}} \longrightarrow U_{\text{Rule}}$$

compiles patient features (labs, meds), mapped concepts (via ontology), and key imaging findings into a rule base that ranks acuity and suggests next actions.

22 Many Other Philosophy

Many other forms of philosophy are also known, and several examples are introduced below. These too are indispensable areas of philosophical research, and they hold promise for extension into arbitrary domains.

Definition 22.1 (Analytic Philosophy). (cf. [205–207]) An *analytic-philosophy structure* is a tuple $\mathbf{AP} = (\mathcal{L}, \vdash, \text{Mod}, \models)$ with a formal language $\mathcal{L}$, a proof relation $\vdash$, a class of models Mod , and satisfaction $\models \subseteq \text{Mod} \times \text{Sent}(\mathcal{L})$. Its core aim is to compare $\vdash$ and semantic consequence $\models$ (soundness & completeness) where $\Gamma \vdash \varphi \iff (\forall \mathfrak{M} \in \text{Mod}) (\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \varphi)$.

Example 22.2 (Analytic Philosophy: Warranty eligibility screening). Let $\mathcal{L}$ contain unary predicates R (has receipt), W (within 1 year), D (manufacturing defect), E (eligible). Proof rule in $\vdash$: $R(x) \wedge W(x) \wedge D(x) \rightarrow E(x)$. A model $\mathfrak{M} \in \text{Mod}$ is a customer case interpreting these predicates as booleans; $\mathfrak{M} \models \varphi$ is usual truth. Then from premises $\Gamma = \{R(c), W(c), D(c)\}$ we derive $\Gamma \vdash E(c)$ and, by soundness, for all $\mathfrak{M}$, if $\mathfrak{M} \models \Gamma$ then $\mathfrak{M} \models E(c)$. This formalizes a sound rule-based warranty triage.

Definition 22.3 (Continental Philosophy). (cf. [208–210]) A *hermeneutic structure* is $\mathbf{HC} = (T, C, M, U)$ where T is a set of texts, C contexts, M meanings, and $U : (T \rightarrow M) \times C \rightarrow (T \rightarrow M)$ an update operator. A *reading* is a fixed point $\mu : T \rightarrow M$ with $\mu = U(\mu, c)$ for relevant $c \in C$; stability across context families formalizes historically situated understanding.

Example 22.4 (Continental Philosophy (Hermeneutics): Corporate values across eras). Let $T = \{\text{mission_statement}\}$, contexts $C = \{\text{growth, recession}\}$, meanings M be mappings from key phrases to commitments (e.g. “customer first” $\mapsto$ *lenient refund*). The update $U : (T \rightarrow M) \times C \rightarrow (T \rightarrow M)$ revises a reading by re-weighting themes under a context (e.g. in recession, “sustainability” emphasizes *repair* over *replace*). A *reading* μ with $\mu = U(\mu, c)$ for a family of contexts captures a stable, historically situated interpretation.

Definition 22.5 (Pragmatist Philosophy). (cf. [211–213]) Fix states S , actions Act , policies Π , transition/likelihood P , and utility $U : S \times \text{Act} \rightarrow \mathbb{R}$. For propositions a, b , define *pragmatic consequence* $a \Rightarrow_{\text{use}} b$ iff for all $\pi \in \Pi$, the expected utility under adopting $a \wedge b$ weakly dominates that under a alone: $\mathbb{E}_{\pi}[U \mid a \wedge b] \geq \mathbb{E}_{\pi}[U \mid a]$, with strict inequality in some admissible task.

Example 22.6 (Pragmatist Philosophy: Choosing mobile order at a café). States S encode queue lengths; actions $\text{Act} = \{\text{walk-in, mobile}\}$; policies Π are wait-time heuristics; utility $U = -(\text{minutes waited})$. Let $a =$ “I have the café app”, $b =$ “use mobile order”. If for all $\pi \in \Pi$ the expected wait satisfies $\mathbb{E}_{\pi}[U \mid a \wedge b] \geq \mathbb{E}_{\pi}[U \mid a]$ (strict in peak hours), then $a \Rightarrow_{\text{use}} b$ holds: adopting b *works better* in practice.

Definition 22.7 (Phenomenological Philosophy). (cf. [214, 215]) A *phenomenological structure* is $\mathbf{Ph} = (X, O, I, r, \mathcal{V})$ with experiences X , intentional objects O , intentionality $I \subseteq X \times O$, an epoché $r : X \rightarrow X$ (idempotent), and a group $\mathcal{V}$ of eidetic variations acting on X . An *invariant* is a property of (X, I) preserved by all $v \in \mathcal{V}$ after reduction r .

Example 22.8 (Phenomenological Philosophy: Recognizing a mug). Experiences X are sense-data frames of a mug under varying viewpoints/lighting; objects $O = \{\text{mug}\}$; intentionality $I \subseteq X \times O$ relates each frame to “mug”. Epoché $r : X \rightarrow X$ brackets background (brand, owner), retaining appearance. Let $\mathcal{V}$ be rotations/illumination changes. The invariant “graspable hollow vessel” (property of (X, I)) persists under all $v \in \mathcal{V}$ after reduction r , capturing the *eidōs*.

Definition 22.9 (Existential Philosophy). (cf. [216, 217]) Let Ag be agents, S states, Act actions. A *choice system* is $\mathbf{EX} = (\text{Ag}, S, \text{Act}, \text{Choose}, \text{Auth}, C)$ with $\text{Choose} : \text{Ag} \times S \rightarrow \mathcal{P}(\text{Act})$, an authenticity predicate $\text{Auth} \subseteq \text{Ag} \times (\text{Act})^*$, and responsibility cost $C : (\text{Act})^* \rightarrow \mathbb{R}_{\geq 0}$. A history h is *existentially warranted* if $\text{Auth}(a, h)$ holds and h minimizes C among available choices.

Example 22.10 (Existential Philosophy: Job offer and authenticity). Agents $\text{Ag} = \{\text{Alice}\}$; state $S = \{\text{offer}\}$; actions $\text{Act} = \{\text{accept, decline, negotiate, retrain}\}$. $\text{Choose}(\text{Alice}, \text{offer})$ lists feasible moves. $\text{Auth}(\text{Alice}, h)$ holds when a history h expresses her core project (e.g. “build ethical AI”). Responsibility cost C penalizes foreseeable harm/time-waste. If $h^* = (\text{decline, retrain})$ is authentic and minimizes C among available histories, then h^* is existentially warranted in $\mathbf{EX}$.

Definition 22.11 (Process Philosophy). (cf. [218–220]) A *process structure* is $\mathbf{PR} = (E, \preceq, \otimes)$ where E are events, $(E, \preceq)$ a causal partial order, and $\otimes$ a partial, associative composition respecting causality ($x \preceq y, y \preceq z \Rightarrow x \preceq x \otimes y \preceq z$ when defined). A *stable entity* is a down-closed subset $D \subseteq E$ closed under $\otimes$.

Example 22.12 (Hospital triage as a real-world Process Philosophy case). Consider an emergency department. Let E be the set of event tokens for a single patient:

$$E = \{\text{Arrival, Triage, Vitals, LabOrder, BloodDraw, Imaging, Diagnosis, Treatment, Discharge}\}.$$

Define the causal partial order $\preceq$ by medical precedence (e.g., $\text{Arrival} \preceq \text{Triage} \preceq \text{Diagnosis} \preceq \text{Treatment}$), and let $\otimes$ be partial composition that concatenates adjacent, causally compatible events into a subprocess (e.g., $\text{Triage} \otimes \text{Vitals}$ is defined, while $\text{Imaging} \otimes \text{Arrival}$ is not). A *stable entity* for “minor trauma care” is the down-closed set $D = \{\text{Arrival, Triage, Vitals, Diagnosis, Treatment, Discharge}\}$, which is closed under $\otimes$. This models care as *organized becoming*: identities (care pathways) are structured patterns of events rather than static substances.

Definition 22.13 (Experimental Philosophy). (cf. [221–223]) An *x-phi protocol* is $\mathbf{XP} = (\text{Pop}, \mathcal{T}, H, \text{Test}, \alpha)$: population Pop , tasks $\mathcal{T}$, hypotheses H , a statistical test Test , and threshold $\alpha \in (0, 1)$. Data D *support* H when $\text{Test}(D, H) \leq \alpha$, thereby constraining conceptual claims by observed judgments.

Example 22.14 (Intentionality judgments as a real-world Experimental Philosophy case). Fix an x-phi protocol $\mathbf{XP} = (\text{Pop}, \mathcal{T}, H, \text{Test}, \alpha)$:

- Pop : adult participants recruited online (e.g., community sample).
- $\mathcal{T}$: read two vignettes (the “CEO helps” vs. “CEO harms” environment scenarios) and answer “Did the CEO intentionally bring about the outcome?” (yes/no).
- H : *Negative side effects elicit higher intentionality attributions than positive ones.*
- Test : chi-square (or logistic regression) comparing “intentional” response rates across vignettes.
- $\alpha = 0.05$.

Let D be the resulting contingency table. If $\text{Test}(D, H) \leq \alpha$, then the data *support* H , constraining a conceptual claim (about ordinary concepts of intention) by observed judgment patterns.

Definition 22.15 (Computational Philosophy). (cf. [224]) A *computational model* of a concept is $\mathbf{CPu} = (\Sigma, A, C, \leq_m)$, with alphabet Σ , decision problem $A \subseteq \Sigma^*$, complexity class C , and many-one reduction $\leq_m$. The concept is *tractable* if $\chi_A \in C$; comparative content is analyzed via completeness/hardness under $\leq_m$.

Example 22.16 (Computational Philosophy: Validity vs. hardness in everyday tasks). **(a) Tractable check: ISBN–13 validation.** Alphabet $\Sigma = \{0, \dots, 9\}$. Decision set $A = \{\text{strings } x \in \Sigma^{13} \mid \text{ISBN–13 checksum holds}\}$. Compute $\chi_A(x)$ in linear time by the rule $\sum_{i=1}^{12} w_i x_i + x_{13} \equiv 0 \pmod{10}$ with $w_i \in \{1, 3\}$ alternating, so $\chi_A \in C = \mathbf{P}$ (tractable).

(b) Hard planning: One-day delivery with time windows. Instance encoded over Σ describing customers, travel times, a single truck, and a deadline L ; decision set $A' = \{\text{instances admitting a route of length } \leq L \text{ meeting all time windows}\}$. There is a many-one reduction $\text{TSP} \leq_m A'$ (embed a TSP metric and tight windows), so A' is NP-hard under $\leq_m$; feasibility screening is computationally demanding.

Definition 22.17 (Neuro-Philosophy). (cf. [225]) A *neuro-implementation frame* is $\mathbf{NP} = (N, C, \pi, D)$, with neural states N , cognitive states C , a realization map $\pi : N \rightarrow C$, and neural data $D \subseteq N$. A hypothesis $\theta \subseteq C$ is *neurally admissible* if $\pi[D] \subseteq \theta$ and θ is realizable by π on relevant tasks.

Example 22.18 (Neuro-Philosophy: Admissible hypothesis from neural data). Let N be EEG epochs, $C = \{\text{cognitive states}\}$ with labels $\{\text{face-recognition, non-face}\}$. Let $\pi : N \rightarrow C$ be a trained decoder that uses the N170 component (negative peak near ~ 170 ms at occipito-temporal sites) to classify each epoch. For a recognition task, take $D \subset N$ the trials with known faces. The hypothesis $\theta = \{\text{face-recognition}\} \subset C$ is *neurally admissible* when $\pi[D] \subseteq \theta$ (all known-face trials decoded as recognition) and the same π realizes θ across held-out sessions; then θ is supported by neural implementation.

Definition 22.19 (Eco-Philosophy). (cf. [226, 227]) Let X be ecological states and $\{F_p : X \rightarrow X\}_{p \in \text{Pol}}$ policy-indexed dynamics. Given a family of *sustainability sets* $\mathcal{S} \subseteq \mathcal{P}(X)$, a policy p is *sustainable* on $S \in \mathcal{S}$ when $F_p[S] \subseteq S$. Normative evaluation prefers policies sustaining designated S .

Example 22.20 (Eco-Philosophy: Sustainable fishery under a simple harvest policy). Let ecological state x_t be biomass (tons). For policy p with constant effort E , use the discrete logistic-with-harvest map

$$F_p(x) = x + r x \left(1 - \frac{x}{K}\right) - q E x,$$

with growth $r = 0.5$, carrying capacity $K = 100$, catchability $q = 0.01$. Define a sustainability set $S = [40, 100] \subset \mathbb{R}_{\geq 0}$. For $x \in S$ the worst-case lower bound is at $x = 40$:

$$F_p(40) = 40 + 0.5 \cdot 40 \left(1 - \frac{40}{100}\right) - 0.01 E \cdot 40 = 52 - 0.4 E.$$

To keep $F_p(40) \geq 40$ we need $E \leq 30$. At $x = 100$, $F_p(100) = 100 - 0.01 E \cdot 100 = 100 - E \leq 100$ and ≥ 40 if $E \leq 60$. Thus for all $x \in S$, if $E \leq 30$ then $F_p[S] \subseteq S$: policy p with effort $E \leq 30$ is sustainable on S .

23 Combined Thought of Structure/Philosophy

This section, as a reference, considers the procedures of the universal merge of two Structure types and the universal merge of two Philosophy types. Looking back at the main purpose of this paper, the main aim is to demonstrate that concepts in any domain can be extended by means of the approaches introduced here. From this standpoint, the author believes that it is possible to examine examples of application procedures to various fields.

Procedure 1 (Universal merge of two Structure types (Any $\rightarrow$ Any)). **Goal.** Given two arbitrary Structure types on carriers H_1, H_2 with operations

$$S_1 = (H_1, \{f_j : F_j(H_1)^{m_j} \rightarrow G_j(H_1)\}), \quad S_2 = (H_2, \{g_k : E_k(H_2)^{n_k} \rightarrow D_k(H_2)\}),$$

construct a single merged Structure on a chosen state space S and certify its well-formedness.

Data functors available. Identity Id, powerset $\mathcal{P}$, finite multiset $\mathcal{M}$. (Other endofunctors can be added if you provide the two bridges below.)

Bridges (canonical, monotone).

$$\iota_{\mathcal{M}} : \mathcal{P}(H) \rightarrow \mathcal{M}(H), \quad \iota_{\mathcal{M}}(X) = \sum_{x \in X} \delta_x,$$

$$\rho : \mathcal{M}(H) \rightarrow \mathcal{P}(H), \quad \rho(\mu) = \text{supp}(\mu).$$

When $S = \mathcal{P}(H)$, interpret $k \odot X = \emptyset$ for $k = 0$ and $= X$ for $k > 0$.

Output. A merged Structure

$$\text{Merge}(S_1, S_2; S) = (S, \{\widehat{f_j} : S^{m_j} \rightarrow S\}_j \cup \{\widehat{g_k} : S^{n_k} \rightarrow S\}_k),$$

with $S \in \{\mathcal{P}(H), \mathcal{M}(H)\}$ over a unified carrier H .

Steps.

S.1 Unify carriers. Choose a common H and embeddings $\eta_1 : H_1 \hookrightarrow H$, $\eta_2 : H_2 \hookrightarrow H$ (e.g. identify, pushout–glue, or disjoint tag). Replace f_j, g_k by η –transported versions on H .

S.2 Choose the state space S . Typical: $S = \mathcal{M}(H)$ (tracks multiplicity/weights) or $S = \mathcal{P}(H)$ (set-valued).

S.3 Fix accumulation on S . On $\mathcal{P}(H)$ use $\boxplus = \cup$, neutral $0_S = \emptyset$; on $\mathcal{M}(H)$ use pointwise sum $\oplus$, neutral $\mathbf{0}$, and scaling $k \odot \mu$ (multiplicity).

S.4 Accessors from S to H .

$$\text{base}_S(s) = \begin{cases} s, & S = \mathcal{P}(H), \\ \text{supp}(s), & S = \mathcal{M}(H), \end{cases} \quad w_S(s)(x) = \begin{cases} \mathbf{1}_{x \in s}, & S = \mathcal{P}(H), \\ s(x), & S = \mathcal{M}(H). \end{cases}$$

S.5 Realize outputs into S . For $Y \in \{H, \mathcal{P}(H), \mathcal{M}(H)\}$ define $\text{Real}_Y^S : Y \rightarrow S$ by

$$\text{Real}_H^{\mathcal{P}}(x) = \{x\}, \quad \text{Real}_H^{\mathcal{M}}(x) = \delta_x, \quad \text{Real}_{\mathcal{P}}^{\mathcal{P}} = \text{Id}, \quad \text{Real}_{\mathcal{P}}^{\mathcal{M}} = \iota_{\mathcal{M}}, \quad \text{Real}_{\mathcal{M}}^{\mathcal{M}} = \text{Id}, \quad \text{Real}_{\mathcal{M}}^{\mathcal{P}} = \rho.$$

S.6 Lift each source operation to S . For $f : F(H)^m \rightarrow G(H)$ define $\widehat{f} : S^m \rightarrow S$ by

$$\widehat{f}(s_1, \dots, s_m) := \bigsqcup_{(x_1, \dots, x_m) \in \prod_{i=1}^m \text{base}_S(s_i)} \left(\min_{1 \leq i \leq m} w_S(s_i)(x_i) \right) \odot \text{Real}_G^S(f(x_1, \dots, x_m)),$$

where $\bigsqcup/\odot$ are from **S.3**. Apply this to every f_j and g_k to obtain $\widehat{f}_j, \widehat{g}_k$.

S.7 Reduction checks (atoms). For $x_1, \dots, x_m \in H$ and the unit elements

$$\epsilon_S(x) = \begin{cases} \{x\}, & S = \mathcal{P}(H), \\ \delta_x, & S = \mathcal{M}(H), \end{cases} \Rightarrow \widehat{f}(\epsilon_S(x_1), \dots, \epsilon_S(x_m)) = \text{Real}_G^S(f(x_1, \dots, x_m)).$$

S.8 Well-formedness & laws. Prove the items in Procedure 2.

Procedure 2 (Validity checklist for Procedure 1). For all lifted operators $\widehat{f}, \widehat{g}$ built by **S.6**:

V.1 Type & finiteness. Finite inputs yield finite sums; hence $\widehat{f}, \widehat{g} : S^m \rightarrow S$ are total.

V.2 Monotonicity. If $s_i \subseteq s'_i$ (sets) or $s_i \preceq s'_i$ (multisets) then $\widehat{f}(\vec{s}) \subseteq \widehat{f}(\vec{s}')$ (resp. $\preceq$); follows from $\min, \bigsqcup$ monotonicity.

V.3 Atomic reduction. The displayed equality in **S.7**.

V.4 Law inheritance (optional). If f is commutative/associative in each coordinate and S -accumulation is commutative/associative, then $\widehat{f}$ inherits those laws (expand sums and regroup).

Example 23.1 (Specialization: MultiStructure + HyperStructure). Take $S = \mathcal{M}(H)$. If $f = \#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ and $\circ : H^2 \rightarrow \mathcal{P}(H)$, then Procedure 1 yields

$$\widehat{\#}^{(m)}(\vec{\mu}) = \bigoplus_{\vec{x} \in \prod \text{supp}(\mu_i)} \left(\min_i \mu_i(x_i) \right) \odot \#^{(m)}(\vec{x}), \quad (\mu \blacklozenge \nu) = \bigoplus_{x \in \text{supp} \mu, y \in \text{supp} \nu} \min\{\mu(x), \nu(y)\} \odot \iota_{\mathcal{M}}(x \circ y).$$

Atomic reduction holds: $\widehat{\#}^{(m)}(\delta_{x_1}, \dots, \delta_{x_m}) = \#^{(m)}(\vec{x})$ and $\delta_x \blacklozenge \delta_y = \iota_{\mathcal{M}}(x \circ y)$.

Procedure 3 (Universal merge of two *philosophical* systems (Any $\rightarrow$ Any)). **Goal.** Merge an arbitrary *Philosophy/philosophy* pair into a single system with well-defined consequence.

Inputs.

- Two closure-bearing systems

$$\mathcal{P}_1 = (Q_1, A_1, \text{adr}_1, \text{Cn}_1), \quad \mathcal{P}_2 = (Q_2, A_2, \text{adr}_2, \text{Cn}_2),$$

where each $\text{Cn}_i : \mathcal{P}(A_i) \rightarrow \mathcal{P}(A_i)$ is extensive, monotone, idempotent (Tarskian).

- *Bridges* (monotone translations) $\tau_{12} : \mathcal{P}(A_1) \rightarrow \mathcal{P}(A_2)$ and $\tau_{21} : \mathcal{P}(A_2) \rightarrow \mathcal{P}(A_1)$.
- (Optional) Question unifiers $u_i : Q_i \rightarrow Q^*$ (to identify synonymous questions).

Output. A merged system $\mathcal{P}_{\bowtie} = (Q^*, A_1 \sqcup A_2, \text{adr}^*, \text{Cn}_{\bowtie})$.

Steps.

P.1 Unify questions. Set $Q^* := Q_1 \sqcup Q_2$ or use u_i to coalesce into a common Q^* . Define $\text{adr}^*|_{A_i} := u_i \circ \text{adr}_i$ (with $u_i = \text{inj}$ if no unifier).

P.2 Define the joint 1-step operator on $A := A_1 \sqcup A_2$. For $X \subseteq A$, write $X_i = X \cap A_i$ and put

$$F(X) := \text{Cn}_1(X_1) \cup \text{Cn}_2(X_2) \cup \tau_{12}(X_1) \cup \tau_{21}(X_2).$$

P.3 Iterate to fixed point (effective definition).

$$X_0 := \Gamma \subseteq A, \quad X_{t+1} := F(X_t), \quad \text{Cn}_{\text{b}\text{d}}(\Gamma) := \bigcup_{t \geq 0} X_t.$$

P.4 Acceptable answers. For $q \in Q^*$,

$$\text{Ans}_{\text{b}\text{d}}(q \mid \Gamma) := \text{Cn}_{\text{b}\text{d}}(\Gamma) \cap (\text{adr}^*)^{-1}(\{q\}).$$

Why this is a closure (sketch). F is monotone (union of monotone maps), hence the Kleene sequence X_t is increasing; extensivity follows from $X_0 \subseteq X_1$; idempotence holds because $F(\text{Cn}_{\text{b}\text{d}}(\Gamma)) = \text{Cn}_{\text{b}\text{d}}(\Gamma)$.

Example 23.2 (Plugging concrete kinds (*any* $\rightarrow$ *any*)). If $\mathcal{P}_1$ is fuzzy (keep only α -cuts) and $\mathcal{P}_2$ is classical, let $\tau_{12}(X)$ = rules translating A_1 claims to A_2 , τ_{21} analogously. Then Procedure 3 produces a crisp merged closure; replacing $\mathcal{P}(A_i)$ by a complete lattice of states $L_i^{A_i}$ and $\cup$ by lattice joins yields the same fixed-point construction (pointwise), with monotone τ_{ij} .

24 Conclusion

The present paper provided reflections on Philosophy and on various forms of Structure. Here we restate the primary purpose of this work: the main aim is to demonstrate that concepts in any domain can be extended by means of the approaches introduced here. We also expect future research to advance toward quantitative analyses and algorithmic designs supported by computer experiments.

Funding

No external funding was received for this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

Acknowledgments

We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

References

- [1] Keekok Lee. *Philosophy and revolutions in genetics: Deep science and deep technology*. Springer, 2016.
- [2] Lauri Koskela et al. *Application of the new production philosophy to construction*, volume 72. Stanford university Stanford, 1992.
- [3] Jenny Teichman. *The Mind and the Soul: an Introduction to the Philosophy of Mind*. Routledge, 2014.
- [4] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
- [5] David Blackwell. On a class of probability spaces. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Contributions to Probability Theory*, volume 3, pages 1–7. University of California Press, 1956.
- [6] John M Chambers and Trevor J Hastie. Statistical models. In *Statistical models in S*, pages 13–44. Routledge, 2017.
- [7] Teturo Inui, Yukito Tanabe, and Yosataka Onodera. *Group theory and its applications in physics*, volume 78. Springer Science & Business Media, 2012.
- [8] Robert Wisbauer. *Foundations of module and ring theory*. Routledge, 2018.
- [9] OR Dehghan. Linear functionals on hypervector spaces. *Filomat*, 34(9):3031–3043, 2020.
- [10] Sehie Park. Fixed point theorems in hyperconvex metric spaces. *Nonlinear Analysis-theory Methods & Applications*, 37:467–472, 1999.
- [11] Reinhard Diestel. Graduate texts in mathematics: Graph theory.
- [12] Anil Nerode. Linear automaton transformations. *Proceedings of the American Mathematical Society*, 9(4):541–544, 1958.
- [13] Russell Vane. Advances in hypergame theory. In *Workshop on Game Theoretic and Decision Theoretic Agents?Conference on Autonomous Agents and Multi-Agent Systems*, 2006.
- [14] Takaaki Fujita and Florentin Smarandache. A unified framework for u -structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.
- [15] Takaaki Fujita. How to represent $a \rightarrow b \rightarrow \dots \rightarrow z$: From curried functions and hyperfunctions to curried structures and hyperstructures, and more, 2025.
- [16] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.
- [17] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.
- [18] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.
- [19] Edwin T. Jaynes, Thomas Boyde Grandy, Ray Smith, Tom Lored, Myron Tribus, John Skilling, and G Larry Bretthorst. Probability theory: the logic of science. *The Mathematical Intelligencer*, 27:83, 2005.
- [20] J. F. C. Kingman and William Feller. An introduction to probability theory and its applications. *Biometrika*, 130:430–430, 1958.
- [21] Jiqian Chen, Jun Ye, and Shigui Du. Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry*, 9:208, 2017.
- [22] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.
- [23] Reiner Lenz and Alan C. Bovik. Group theory. 2007.
- [24] Dietrich Burde. Group theory. *Computers, Rigidity, and Moduli*, 2019.
- [25] Bo Stenstrom. *Rings of quotients: an introduction to methods of ring theory*, volume 217. Springer Science & Business Media, 2012.
- [26] OR Dehghan, R Ameri, and HA Ebrahimi Aliabadi. Some results on hypervector spaces. *Italian Journal of Pure and Applied Mathematics*, 41:23–41, 2019.

-
- [27] K Sameena et al. Fuzzy matroids from fuzzy vector spaces. *South East Asian Journal of Mathematics and Mathematical Sciences*, 17(03):381–390, 2021.
 - [28] Ahmed Hatip, Necati Olgun, et al. On the concepts of two-fold fuzzy vector spaces and algebraic modules. *Journal of Neutrosophic and Fuzzy Systems*, 7(2):46–52, 2023.
 - [29] Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33):12, 2005.
 - [30] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.
 - [31] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.
 - [32] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
 - [33] Zhe Hou. Automata theory and formal languages. *Texts in Computer Science*, 2021.
 - [34] Borzoo Bonakdarpour and Sarai Sheinvald. Automata for hyperlanguages. *arXiv preprint arXiv:2002.09877*, 2020.
 - [35] Nicholas Kovach, Alan S. Gibson, and Gary B. Lamont. Hypergame theory: A model for conflict, misperception, and deception. 2015.
 - [36] James Thomas House and George V. Cybenko. Hypergame theory applied to cyber attack and defense. In *Defense + Commercial Sensing*, 2010.
 - [37] Gulay Oguz and Bijan Davvaz. Soft topological hyperstructure. *J. Intell. Fuzzy Syst.*, 40:8755–8764, 2021.
 - [38] Souzana Vougioukli. Helix hyperoperation in teaching research. *Science & Philosophy*, 8(2):157–163, 2020.
 - [39] Souzana Vougioukli. Helix-hyperoperations on lie-santilli admissibility. *Algebras Groups and Geometries*, 2023.
 - [40] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.
 - [41] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
 - [42] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.
 - [43] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.
 - [44] Takaaki Fujita. Chemical hyperstructures, superhyperstructures, and shv-structures: Toward a generalized framework for hierarchical chemical modeling. 2025.
 - [45] Peter Van Inwagen. *Metaphysics*. Routledge, 2024.
 - [46] Martin Heidegger. *Metaphysics*. In *Journey into Philosophy*, pages 179–185. Routledge, 2016.
 - [47] Matthias Steup and Ram Neta. *Epistemology*. 2005.
 - [48] Earl Conee and Richard Feldman. *Evidentialism: Essays in epistemology*. Clarendon Press, 2004.
 - [49] Jean Hampton. *Political philosophy*. Routledge, 2018.
 - [50] Will Kymlicka. *Contemporary political philosophy: An introduction*. oxford: oxford University Press, 2002.
 - [51] Aloysius P Martinich. Philosophy of language. In *Philosophy of meaning, knowledge and value in the twentieth century*, pages 8–26. Routledge, 2012.
 - [52] John Heil. *Philosophy of mind: A contemporary introduction*. Routledge, 2019.
 - [53] Philipp Frank, Carl Gustav Hempel, Paul Oppenheim, and Kurt H Wolff. *Philosophy of science*. Prentice-Hall Englewood Cliffs, NJ, 1957.
 - [54] Stephen Toulmin. *The philosophy of science*. Hutchinson London, 1953.
 - [55] Leon Horsten. *Philosophy of mathematics*. 2007.
 - [56] Paul Benacerraf and Hilary Putnam. *Philosophy of mathematics: Selected readings*. Cambridge University Press, 1983.
 - [57] Roberto Torretti. *The philosophy of physics*. Cambridge University Press, 1999.
 - [58] Thomas A Brody. *The philosophy behind physics*. Springer Science & Business Media, 2012.
 - [59] Elliott Sober. *Philosophy of biology*. Routledge, 2018.
 - [60] Ernst Mayr. *Toward a new philosophy of biology: Observations of an evolutionist*. Number 211. Harvard University Press, 1988.
 - [61] Don Ihde. Philosophy of technology. In *Philosophical problems today: World and worldhood*, pages 91–108. Springer, 2004.
 - [62] Maarten Franssen, Gert-Jan Lokhorst, and Ibo Van de Poel. *Philosophy of technology*. 2009.
 - [63] Thomas Dick. *The philosophy of religion*, volume 3. EC & J. Biddle, 1845.
 - [64] John Morreall. Philosophy and religion. *The Primer of Humour Research, Berlin and New York: Mouton de Gruyter*, pages 211–242, 2008.
 - [65] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. 2025.
 - [66] Takaaki Fujita. Meta-fuzzy graph, meta-neutrosophic graph, meta-digraph, and meta-multigraph with some applications. 2025.
 - [67] Takaaki Fujita. Metafuzzy, metaneutrosophic, metasoft, and metarough set. 2025.
 - [68] Takaaki Fujita. Metastructure, meta-hyperstructure, and meta-superhyperstructure, 2025. Preprint.
 - [69] Søren Overgaard, Paul Gilbert, and Stephen Burwood. *An introduction to metaphilosophy*. Cambridge University Press, 2013.
 - [70] Nicholas Joll. *Metaphilosophy*. 2010.
 - [71] Vadim V Vasilyev. Metaphilosophy: History and perspectives. *Epistemology & Philosophy of Science*, 56(2):6–18, 2019.
 - [72] Jelscha Schmid. *The methods of metaphilosophy*. Klostermann, 2022.
 - [73] Bijan Davvaz and Thomas Vougiouklis. *Walk Through Weak Hyperstructures, A: Hv-structures*. World Scientific, 2018.
 - [74] Piergiulio Corsini and Violeta Leoreanu. *Applications of hyperstructure theory*, volume 5. Springer Science & Business Media, 2013.
 - [75] THOMAS Vougiouklis. The fundamental relation in hyperrings. the general hyperfield. In *Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990)*, *World Scientific*, pages 203–211. World Scientific, 1991.
 - [76] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.
 - [77] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.
 - [78] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.
 - [79] Yuxin Wang, Quan Gan, Xipeng Qiu, Xuanjing Huang, and David Wipf. From hypergraph energy functions to hypergraph neural networks. In *International Conference on Machine Learning*, pages 35605–35623. PMLR, 2023.

-
- [80] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.
 - [81] Ivo G Rosenberg. An algebraic approach to hyperalgebras. In *Proceedings of 26th IEEE International Symposium on Multiple-Valued Logic (ISMVL'96)*, pages 203–207. IEEE, 1996.
 - [82] Kostaq Hila, Serkan Onar, Bayram Ali Ersoy, and Bijan Davvaz. On generalized intuitionistic fuzzy subhyperalgebras of boolean hyperalgebras. *Journal of Inequalities and Applications*, 2013:1–15, 2013.
 - [83] Florentin Smarandache. Extension of hyperalgebra to superhyperalgebra and neutrosophic superhyperalgebra (revisited). In *International Conference on Computers Communications and Control*, pages 427–432. Springer, 2022.
 - [84] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.
 - [85] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.
 - [86] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.
 - [87] Takaaki Fujita. Rethinking strategic perception: Foundations and advancements in hypergame theory and superhypergame theory. *Prospects for Applied Mathematics and Data Analysis*, 4(2):01–14, 2024.
 - [88] Junjie Jiang, Yu-Zhong Chen, Zi-Gang Huang, and Ying-Cheng Lai. Evolutionary hypergame dynamics. *Physical Review E*, 98(4):042305, 2018.
 - [89] Takaaki Fujita. Hyperalgorithms & superhyperalgorithms: A unified framework for higher-order computation. *Prospects for Applied Mathematics and Data Analysis*, 4(1):36–49, 2024.
 - [90] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.
 - [91] Michael Rathjen. Constructive zermelo-fraenkel set theory, power set, and the calculus of constructions. In *Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf*, pages 313–349. Springer, 2012.
 - [92] Melody Mae Cabigting Lunar and Renson Aguilar Robles. Characterization and structure of a power set graph. *International Journal of Advanced Research and Publications*, 3(6):1–4, 2019.
 - [93] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.
 - [94] Takaaki Fujita. Superhypermagma, lie superhypergroup, quotient superhypergroups, and reduced superhypergroups. *International Journal of Topology*, 2(3):10, 2025.
 - [95] Adel Al-Odhari. A brief comparative study on hyperstructure, super hyperstructure, and n-super superhyperstructure. *Neutrosophic Knowledge*, 6:38–49, 2025.
 - [96] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. Technical report, Center for Open Science, 2024.
 - [97] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.
 - [98] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.
 - [99] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.
 - [100] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.
 - [101] Florentin Smarandache. The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures. *Systems Assessment and Engineering Management*, 2:19–22, 2024.
 - [102] Takaaki Fujita. Hierarchical uncertainty modeling via (m, n)-superhyperuncertain and (h, k)-ary (m, n)-superhyperuncertain sets: Unified extensions of fuzzy, neutrosophic, and plithogenic set theories. *Preprints*, 2025.
 - [103] Takaaki Fujita. Introduction for structure, hyperstructure, superhyperstructure, multistructure, iterative multistructure, treestructure, and foreststructure. 2025.
 - [104] Takaaki Fujita. Iterative multistructure and curried iterative multistructure: Graph, function, chemical structure, and beyond. 2025.
 - [105] WB Vasanthi Kandasamy, K Ilanthenral, and Florentin Smarandache. *Subset Vertex Multigraphs and Neutrosophic Multigraphs for Social Multi Networks*. Infinite Study, 2019.
 - [106] Ryan C. Bunge, M. K. Chwee, Andrew Michael Wokingham Cooper, Saad I. El-Zanati, Katlyn Kennedy, Dan P. Roberts, and C. C. Wilson. The spectrum for a multigraph on 4 vertices and 7 edges. 2018.
 - [107] Ryan C. Bunge, Joel Jeffries, Julie Kirkpatrick, Dan P. Roberts, and A. L. Sickman. Spectrum for multigraph designs on four vertices and six edges. 2017.
 - [108] Serge Grigorieff and Pierre Valarcher. Evolving multialgebras unify all usual sequential computation models. *arXiv preprint arXiv:1001.2160*, 2010.
 - [109] Mahdie Haddadi. Fuzzy acts over fuzzy semigroups and sheaves. *Iranian Journal of Fuzzy Systems*, 2014.
 - [110] Ambalal V Patel. Analytical structures and analysis of fuzzy pd controllers with multifuzzy sets having variable cross-point level. *Fuzzy sets and Systems*, 129(3):311–334, 2002.
 - [111] Jun Ye, Shigui Du, and Rui Yong. Multifuzzy cubic sets and their correlation coefficients for multicriteria group decision-making. *Mathematical Problems in Engineering*, 2021(1):5520335, 2021.
 - [112] Kholood Mohammad Alsager. Decision-making framework based on multineutrosophic soft rough sets. *Mathematical Problems in Engineering*, 2022(1):2868970, 2022.
 - [113] K Reji Kumar and SA Naisal. Interior exterior and boundary of fuzzy soft multi topology in decision making. In *2016 International Conference on Control, Instrumentation, Communication and Computational Technologies (ICCICCT)*, pages 147–152. IEEE, 2016.
 - [114] Ismail Osmanoglu and Deniz Tokat. Compact soft multi spaces. *European Journal of Pure and Applied Mathematics*, 7(1):97–108, 2014.
 - [115] TK Shinoj and Sunil Jacob John. Intuitionistic fuzzy multisets and its application in medical diagnosis. *World academy of science, engineering and technology*, 6(1):1418–1421, 2012.
 - [116] PA Ejegwa. New operations on intuitionistic fuzzy multisets. *Journal of Mathematics and Informatics*, 3:17–23, 2015.
 - [117] KP Girish and Sunil Jacob John. Rough multisets and information multisystems. *Advances in Decision sciences*, 2011.
 - [118] KP Girish and Sunil Jacob John. On rough multiset relations. *International Journal of Granular Computing, Rough Sets and Intelligent Systems*, 3(4):306–326, 2014.
 - [119] Takaaki Fujita. Fuzzy multisets and iterated fuzzy multisets: Applications in sports culture. 2025.
 - [120] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

-
- [121] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.
 - [122] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.
 - [123] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellig*, pages 305–309, 1994.
 - [124] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.
 - [125] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.
 - [126] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.
 - [127] Sankar Das, Soumitra Poulik, and Ganesh Ghorai. Picture fuzzy φ -tolerance competition graphs with its application. *Journal of Ambient Intelligence and Humanized Computing*, 15(1):547–559, 2024.
 - [128] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.
 - [129] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.
 - [130] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.
 - [131] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.
 - [132] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.
 - [133] Claudio Urrea, John Kern, and Johanna Alvarado. Design and evaluation of a new fuzzy control algorithm applied to a manipulator robot. *Applied sciences*, 10(21):7482, 2020.
 - [134] Xizheng Ke and Danyu Zhang. Fuzzy control algorithm for adaptive optical systems. *Applied optics*, 58(36):9967–9975, 2019.
 - [135] Kein Huat Chua, Yun Seng Lim, and Stella Morris. A novel fuzzy control algorithm for reducing the peak demands using energy storage system. *Energy*, 122:265–273, 2017.
 - [136] Andreas Nürnberger, Detlef Nauck, and Rudolf Kruse. Neuro-fuzzy control based on the nefcon-model: recent developments. *Soft Computing*, 2(4):168–182, 1999.
 - [137] Ayad Hendalianpour and Jafar Razmi. Customer satisfaction measurement using fuzzy neural network. *Decis Sci Lett*, 6(2):193–206, 2017.
 - [138] MM Gupta and DH Rao. On the principles of fuzzy neural networks. *Fuzzy sets and systems*, 61(1):1–18, 1994.
 - [139] Hon Keung Kwan and Yaling Cai. A fuzzy neural network and its application to pattern recognition. *IEEE transactions on Fuzzy Systems*, 2(3):185–193, 1994.
 - [140] Oscar Castillo and Patricia Melin. Interval type-3 fuzzy decision-making in material surface quality control. In *Virtual International Conference on Soft Computing, Optimization Theory and Applications*, pages 157–169. Springer, 2021.
 - [141] Ngoc Bao Tu Nguyen, Guochao Lin, and Thanh-Tuan Dang. A two phase integrated fuzzy decision-making framework for green supplier selection in the coffee bean supply chain. *Mathematics*, 2021.
 - [142] Samanta Bacić, Hrvoje Tomić, Katarina Rogulj, and Goran Andlar. Fuzzy decision-making valuation model for urban green infrastructure implementation. *Energies*, 2024.
 - [143] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.
 - [144] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.
 - [145] XL Xin, FH Xiao, and XF Yang. L-fuzzy algebraic substructure. *Journal of Algebraic Hyperstructures and Logical Algebras*, 4(2):17–35, 2023.
 - [146] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, and Abdallah Al-Husban. An example of two-fold fuzzy algebras based on neutrosophic real numbers. *Neutrosophic Sets and Systems*, 67:169–178, 2024.
 - [147] SA Siddiqui, AQ Ansari, and Shikha Agarwal. A journey through fuzzy philosophy. *Pranjana-The Journal of Management Awareness, India*, 6(2):29–33, 2003.
 - [148] Wayne Tyson. Fuzzy philosophy: A foundation for interneted ecology? *Conservation Ecology*, 5(2), 2002.
 - [149] ŞEN Zekai. Fuzzy philosophy of science. *Journal of Higher Education and Science*, 2(1):020–024, 2012.
 - [150] Z Sen. Fuzzy philosophy of science and education. *Turkish Journal of Fuzzy Systems*, 2(2):77–98, 2011.
 - [151] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.
 - [152] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.
 - [153] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.
 - [154] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
 - [155] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.
 - [156] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.
 - [157] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.
 - [158] Muhammad Akram and Musavarah Sarwar. New applications of m-polar fuzzy competition graphs. *New Mathematics and Natural Computation*, 14(02):249–276, 2018.
 - [159] Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.
 - [160] Heng He. A novel approach to assessing art education teaching quality in vocational colleges based on double-valued neutrosophic numbers and multi-attribute decision-making with tree soft sets. *Neutrosophic Sets and Systems*, 78:206–218, 2025.
 - [161] Qiuyan Zhao and Wentao Li. Incorporating intelligence in multiple-attribute decision-making using algorithmic framework and double-valued neutrosophic sets: Varied applications to employment quality evaluation for university graduates. *Neutrosophic Sets and Systems*, 76:59–78, 2025.
 - [162] Lingling Lin. Double-valued neutrosophic offset for enhancing humanistic competence: An effectiveness study of humanities instruction in vocational college students. *Neutrosophic Sets and Systems*, 88:680–689, 2025.
 - [163] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-

- valued indeterministic set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.
- [164] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14*, 2021, page 11, 2021.
- [165] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.
- [166] Manal Al-Labadi, Shuker Khalil, VR Radhika, K Mohana, et al. Pentapartitioned neutrosophic vague soft sets and its applications. *International Journal of Neutrosophic Science*, (2):64–4, 2025.
- [167] Rakhal Das and Suman Das. Pentapartitioned neutrosophic subtraction algebra. *Neutrosophic Sets and Systems*, 68:89–98, 2024.
- [168] Rama Mallick and Surapati Pramanik. *Pentapartitioned neutrosophic set and its properties*, volume 36. Infinite Study, 2020.
- [169] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.
- [170] Juanjuan Ding, Wenhui Bai, and Chao Zhang. A new multi-attribute decision making method with single-valued neutrosophic graphs. *International Journal of Neutrosophic Science*, 2021.
- [171] M Hamidi and A Borumand Saeid. Accessible single-valued neutrosophic graphs. *Journal of Applied Mathematics and Computing*, 57:121–146, 2018.
- [172] S Narasimman, M Shanmugapriya, R Sundareswaran, Laxmi Rathour, Lakshmi Narayan Mishra, Vinita Dewangan, and Vishnu Narayan Mishra. Identification of influential factors affecting student performance in semester examinations in the educational institution using score topological indices in single valued neutrosophic graphs. *Neutrosophic Sets and Systems*, 75:224–240, 2025.
- [173] Rupkumar Mahapatra, Sovan Samanta, Madhumangal Pal, and Qin Xin. Link prediction in social networks by neutrosophic graph. *Int. J. Comput. Intell. Syst.*, 13:1699–1713, 2020.
- [174] W. B. Vasantha Kandasamy and Florentin Smarandache. Basic neutrosophic algebraic structures and their application to fuzzy and neutrosophic models. *viXra*, 2004.
- [175] Mohammad Abobala. On some neutrosophic algebraic equations. *Journal of New Theory*, 33:26–32, 2020.
- [176] T Nagaiah, L Bhaskar, P Narasimha Swamy, and Said Broumi. A study on neutrosophic algebra. *Neutrosophic Sets Syst*, 50:111–118, 2022.
- [177] Said Broumi, Mohamed Talea, Florentin Smarandache, and Assia Bakali. Decision-making method based on the interval valued neutrosophic graph. *2016 Future Technologies Conference (FTC)*, pages 44–50, 2016.
- [178] Said Broumi, Swaminathan Mohanaselvi, Tomasz Witczak, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Complex fermatean neutrosophic graph and application to decision making. *Decision Making: Applications in Management and Engineering*, 2023.
- [179] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.
- [180] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.
- [181] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.
- [182] Nader Mahmoud Taffach and Ahmed Hatip. A review on symbolic 2-plithogenic algebraic structures. *Galoitica Journal Of Mathematical Structures and Applications*, 5, 2023.
- [183] AAA Agboola and MA Ibrahim. On symbolic plithogenic algebraic structures and hyper structures. *Neutrosophic Sets and Systems*, 56(1):17, 2023.
- [184] Hamiyet Merkepçi and Mohammad Abobala. *On The Symbolic 2-Plithogenic Rings*. Infinite Study, 2023.
- [185] Mohamed Abdel-Basset, Rehab Mohamed, Abdel Nasser H. Zaied, and Florentin Smarandache. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. *Symmetry*, 11:903, 2019.
- [186] Penelope Maddy. Wittgenstein’s anti-philosophy of mathematics. *Wittgenstein’s philosophy of mathematics*, pages 52–72, 1993.
- [187] Alexandre Losev. Wittgenstein, the antiphilosopher, and mathematics. 2014.
- [188] Anderson Nakano. Anti-realism and anti-revisionism in wittgenstein’s philosophy of mathematics. *grazer philosophische studien*, 97(3):451–474, 2020.
- [189] Florentin Smarandache and Madeleine Al-Tahan. Neutroalgebra and antialgebra are generalizations of classical algebras. In *Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras*, pages 1–10. IGI Global, 2022.
- [190] AAA Agboola, MA Ibrahim, and EO Adeleke. *Elementary examination of neutroalgebras and antialgebras viz-a-viz the classical number systems*. Infinite Study, 2020.
- [191] David R Cole. Educational non-philosophy. In *Educational Philosophy and New French Thought*, pages 4–17. Routledge, 2017.
- [192] Simon O’Sullivan. Non-philosophy and art practice (or, fiction as method). *Fiction as Method, Berlin: Sternberg*, 2017.
- [193] Howard Caygill. *A Kant dictionary*. Blackwell Reference., 2020.
- [194] Henry Sidgwick. A criticism of the critical philosophy. *Mind*, 8(31):313–337, 1883.
- [195] Michael A Peters and Tina Besley. Critical philosophy of the postdigital. *Postdigital Science and Education*, 1(1):29–42, 2019.
- [196] Kai Nielsen. Philosophy as critical theory. In *Proceedings and addresses of the American Philosophical Association*, volume 61, pages 89–108. JSTOR, 1987.
- [197] Emily R Lai. Critical thinking: A literature review. *Pearson’s Research Reports*, 6(1):40–41, 2011.
- [198] William Huitt. Critical thinking: An overview. *Educational psychology interactive*, 3(6):34–50, 1998.
- [199] Brooke Noel Moore and Richard Parker. *Critical thinking*. McGraw-Hill, 2012.
- [200] Victor Moberger. Bullshit, pseudoscience and pseudophilosophy. *Theoria*, 86(5):595–611, 2020.
- [201] James Michell Winn. Modern pseudo-philosophy. *Journal of Psychological Medicine and Mental Pathology (London, England: 1875)*, 4(Pt 1):1, 1878.
- [202] RW Maslen. William baldwin and the politics of pseudo-philosophy in tudor prose fiction. *Studies in Philology*, 97(1):29–60, 2000.
- [203] Takaaki Fujita. Hierarchical probability models: Tree and forest structures for treeprobability and forestprobability. *Preprint*, 2025.
- [204] Takaaki Fujita. Structural properties of algebraic treestructures and algebraic foreststructures. *Preprint*, 2025.
- [205] Hanjo Glock. What is analytic philosophy. *Journal for the History of Analytical Philosophy*, 2(2), 2013.
- [206] Anat Biletzki and Anat Matar. The story of analytic philosophy. *Plot and heroes*. Nueva York: Routledge, 1998.
- [207] Tom Sorell and Graham Alan John Rogers. *Analytic philosophy and history of philosophy*. OUP Oxford, 2005.

-
- [208] David West. *Continental philosophy: An introduction*. Polity, 2010.
 - [209] Andrew Cutrofello. *Continental philosophy: A contemporary introduction*. Routledge, 2004.
 - [210] Gary Gutting. *Continental philosophy of science*. John Wiley & Sons, 2008.
 - [211] Robert B Talisse. *A pragmatist philosophy of democracy*. Routledge, 2013.
 - [212] Sami Pihlström. *Toward a pragmatist philosophy of the humanities*. State University of New York Press, 2022.
 - [213] Peter Reason. Pragmatist philosophy and action research: Readings and conversation with richard rorty. *Action Research*, 1(1):103–123, 2003.
 - [214] Tommie Nelms. Phenomenological philosophy and research. *Nursing research using phenomenology: qualitative designs and methods in nursing*. New York: Springer Pub Com, pages 1–23, 2014.
 - [215] Steven M Rosen. Why natural science needs phenomenological philosophy. *Progress in Biophysics and Molecular Biology*, 119(3):257–269, 2015.
 - [216] Paul Tillich. Existential philosophy. *Journal of the History of Ideas*, pages 44–70, 1944.
 - [217] Herbert Spiegelberg and Moritz Geiger. An introduction to existential philosophy. *Philosophy and Phenomenological Research*, 3(3):255–278, 1943.
 - [218] Johanna Seibt. *Process philosophy*. 2012.
 - [219] Nicholas Rescher. *Process philosophy: A survey of basic issues*. University of Pittsburgh Pre, 2012.
 - [220] David Ray Griffin. *Reenchantment without supernaturalism: A process philosophy of religion*. Cornell university press, 2019.
 - [221] Joshua Knobe, Wesley Buckwalter, Shaun Nichols, Philip Robbins, Hagop Sarkissian, and Tamler Sommers. Experimental philosophy. *Annual review of psychology*, 63(1):81–99, 2012.
 - [222] Joshua Knobe. Experimental philosophy. *Philosophy Compass*, 2(1):81–92, 2007.
 - [223] Justin Sytsma and Jonathan Livengood. *The theory and practice of experimental philosophy*. Broadview Press, 2015.
 - [224] Patrick Grim and Daniel Singer. *Computational philosophy*. 2020.
 - [225] Georg Northoff. *Neuro-philosophy and the healthy mind: Learning from the unwell brain*. WW Norton & Company, 2016.
 - [226] Andrzej Papuziński. The idea of philosophy vs. eco-philosophy. *Problemy Ekorozwoju*, 4(1):51–59, 2009.
 - [227] KANU Ikechukwu Anthony. African eco-philosophy: Nature and foundations. *African Eco-Philosophy: Cosmology, Consciousness and the Environment*, 2021.